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A NEW FAMILY OF ESTIMATORS OF THE POPULATION VARIANCE USING INFORMATION ON POPULATION VARIANCE OF AUXILIARY VARIABLE IN SAMPLE SURVEYS

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ABSTRACT

This paper proposes a family of estimators of population variance S_y^2 of the study variable *y* in the presence of known population variance S_x^2 of the auxiliary variable *x*. It is identified that in addition to many, the recently proposed classes of estimators due to Sharma and Singh (2014) and Singh and Pal (2016) are members of the proposed family of estimators. Asymptotic expressions of bias and mean squared error (*MSE*) of the suggested family of estimators have been obtained. Asymptotic optimum estimator (*AOE*) in the family of estimators is identified. Some subclasses of estimators of the proposed family of estimators have been identified along with their properties. We have also given the theoretical comparisons among the estimators discussed in this paper.
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1. Introduction

The problem of estimating the population variance assumes importance in various fields such as industry, agriculture, medical and biological sciences etc. In sample surveys, auxiliary information on the finite population under investigation is quite often available from previous experience, census or administrative databases. It is well known that the auxiliary information in the theory of sampling is used to increase the efficiency of the estimators of the parameters such as mean or total, variance, coefficient of variation etc. Out of many, ratio and regression methods of estimation are good examples in this context. In many

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situations of practical importance, the problem of estimating the population variance S_y^2 of the study variable y deserves special attention. When the population parameters such as population mean, variance, coefficients of skewness and kurtosis of the auxiliary variable are known, several authors including Das and Tripathi (1978), Srivastava and Jhajj (1980), Isaki (1983), Prasad and Singh (1990, 1992), Kadilar and Cingi (2006), Shabbir and Gupta (2007), Gupta and Shabbir (2008), Singh and Solanki (2013a, b), Solanki and Singh(2013), Singh et al. (2013, 2014), Hilal et al. (2014), Sharma and Singh (2014), Solanki et al. (2015), Yadav et al. (2015) and Singh and Pal (2016) etc. have suggested various estimators and studied their properties.

The principal aim of this paper is to suggest a new family of estimators of the population variance S_y^2 of the study variable y using information on population variance S_x^2 of the auxiliary variable x along with its properties under large sample approximation.

Consider a finite population $U=\{U_1, U_2, ..., U_N\}$ of N units. Let y and x be the study and auxiliary variates respectively. We define the following parameters of the variates y and x:

Population mean: $\bar{Y}=N^{-1}\sum_{i=1}^{N} y_i$,

Population mean: $\bar{X}=N^{-1}\sum_{i=1}^{N} x_i$,

Population variance /mean square: $S_y^2=(N-1)^{-1}\sum_{i=1}^{N}(y_i-\bar{Y})^2$,

Population variance /mean square: $S_x^2=(N-1)^{-1}\sum_{i=1}^{N}(x_i-\bar{X})^2$.

Q_i is the i^{th} quartile (i=1, 2, 3) of the auxiliary variable x,

$Q_r=Q_3-Q_1$: the population inter-quartile range of the auxiliary variable x,

$Q_d=(Q_3-Q_1)/2$: the population semi-quartile range of the auxiliary variable x,

$Q_a=(Q_3+Q_1)/2$: the population semi-quartile average of the auxiliary variable x.

It is desired to estimate the population variance S_y^2 of the study variable y when the population variance S_x^2 of the auxiliary variable x is known. For estimating population variance S_y^2, a simple random sample (*SRS*) of size n is drawn without replacement (*WOR*) from the population *U*. The conventional unbiased estimators of the population parameters $(\bar{Y}, S_y^2, \bar{X}, S_x^2)$ are respectively defined by

$$\bar{y} = n^{-1}\sum_{i=1}^{n} y_i \text{ , } s_y^2 = (n-1)^{-1}\sum_{i=1}^{n}(y_i - \bar{y})^2 \text{ , } \bar{x} = n^{-1}\sum_{i=1}^{n} x_i$$

$$\text{and } s_x^2 = (n-1)^{-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \text{ .}$$

When the population variance S_x^2 of the auxiliary variable *x* is known, Isaki (1983) suggested a ratio-type estimator for estimating population variance S_y^2 defined by

$$t_1 = s_y^2\left(\frac{S_x^2}{s_x^2}\right). \tag{1.1}$$

Upadhyaya and Singh (1986) suggested an alternative estimator for S_y^2 as

$$t_2 = s_y^2\left(\frac{s_x^{*2}}{S_x^2}\right), \tag{1.2}$$

where $s_x^{*2} = (NS_x^2 - ns_x^2)/(N-n)$

$= \{(1+g)S_x^2 - gs_x^2\}$

and $g = n/(N-n)$.

Das and Tripathi (1978) suggested a difference-type estimator for the population variance S_y^2 as

$$t_3 = s_y^2 + d(S_x^2 - s_x^2), \tag{1.3}$$

where '*d*' is suitable chosen constant.

Shabbir (2006) suggested a class of estimators of S_y^2 as

$$t_4 = \eta s_y^2 + (1-\eta)s_y^2\left(\frac{s_x^{*2}}{S_x^2}\right), \tag{1.4}$$

where η being suitable chosen constant.

It is well known that the estimator $t_0 = s_y^2$ is an unbiased estimator of S_y^2. The variance *MSE* of s_y^2 under *SRSWOR* to the first degree of approximation is given by

$$Var(t_0) = MSE(t_0) = f\, S_y^4(\lambda_{40} - 1), \tag{1.5}$$

where $\lambda_{40} = \mu_{40}/\mu_{20}^2$, $\mu_{40} = (N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})^4$,

$$\mu_{20} = (N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2 \text{ and } f = \left((1/n) - (1/N)\right).$$

The biases and mean squared errors of the Isaki's (1983) estimator t_1 and Upadhyaya and Singh's (1986) estimator t_2 to the first degree of approximation are, respectively, given by

$$B(t_1) = S_y^2 f(\lambda_{40} - 1)(1 - C), \tag{1.6}$$

$$B(t_2) = -S_y^2 g\, f(\lambda_{04} - 1)C, \tag{1.7}$$

$$MSE(t_1) = S_y^4 f[(\lambda_{40} - 1) + (\lambda_{04} - 1)(1 - 2C)], \tag{1.8}$$

and

$$MSE(t_2) = S_y^4\, f[(\lambda_{40} - 1) + g(\lambda_{04} - 1)(g - 2C)], \tag{1.9}$$

where $\lambda_{pq} = \mu_{pq} / (\mu_{20}^{p/2} \mu_{02}^{q/2})$, $\mu_{pq} = (N-1)^{-1} \sum_{i=1}^{N} (y_i - \bar{Y})^p (x_i - \bar{X})^q$, ($p$, q) being non negative integers; and $C = (\lambda_{22} - 1)/(\lambda_{04} - 1)$.

It is easy to verify that

$$E(t_3) = S_y^2$$

which shows that the difference estimator t_3 is unbiased for S_y^2.

The variance of the difference estimator t_3 to the first degree of approximation is given by

$$MSE(t_3) = Var(t_3) = f\, S_y^4 [(\lambda_{40} - 1) + d(\lambda_{04} - 1)(d - 2C)] \tag{1.10}$$

which is minimum when

$$d = C. \tag{1.11}$$

Thus the resulting minimum MSE of t_3 to the first degree of approximation is given by

$$\begin{aligned} \min.MSE(t_3) &= f\, S_y^4 [(\lambda_{40} - 1) - (\lambda_{04} - 1)C^2] \\ &= f\, S_y^4 (\lambda_{40} - 1)(1 - \rho^{*2}), \end{aligned} \tag{1.12}$$

where $$\rho^* = \frac{Cov(s_y^2, s_x^2)}{\sqrt{Var(s_y^2) Var(s_x^2)}} = \frac{(\lambda_{22} - 1)}{\sqrt{(\lambda_{40} - 1)(\lambda_{04} - 1)}}.$$

To the first degree of approximation, the bias and MSE of the Shabbir's (2006) estimator t_4 are respectively given by

$$B(t_4) = -S_y^2 g\, f(1 - \eta)(\lambda_{04} - 1)C, \tag{1.13}$$

$$MSE(t_4) = S_y^4 f[(\lambda_{40} - 1) + g(1 - \eta)(\lambda_{04} - 1)\{g(1 - \eta) - 2C\}]. \tag{1.14}$$

The $MSE(t_4)$ is minimum when

$$\eta_{opt} = \left(1 - \frac{C}{g}\right). \tag{1.15}$$

Thus the resulting minimum *MSE* of Shabbir (2006) estimator t_4 is given by

$$\min.MSE(t_4) = f\,S_y^4(\lambda_{40} - 1)(1 - \rho^{*2}) \tag{1.16}$$

which equals to the minimum *MSE* of the difference-type estimator t_3 [*i.e.* $\min.MSE(t_4) = \min.MSE(t_3)$].

1.1. Sharma and Singh's (2014) estimators

Sharma and Singh (2014) have suggested three classes of estimators of the population variance S_y^2 as:

$$t_5 = w_1 s_y^2 + w_2(S_x^2 - s_x^{*2}), \tag{1.17}$$

$$t_6 = k_1 s_y^2 + k_2(S_x^2 - s_x^{*2})\left[2 - \left(\frac{s_x^{*2}}{S_x^2}\right)\right], \tag{1.18}$$

and

$$t_7 = m_1 s_y^2\left(\frac{s_x^{*2}}{S_x^2}\right) + m_2(S_x^2 - s_x^{*2}), \tag{1.19}$$

where (w_1, w_2),$(k_1, k_2)$and (m_1, m_2) are suitable chosen scalars such that mean squared errors of t_5, t_6 and t_7 are respectively minimum. We note here that the minimum mean squared errors of the estimators t_5, t_6 and t_7 obtained by Sharma and Singh (2014) are incorrect. Therefore the first objective of the authors of the present paper is to give the correct expressions of the minimum mean squared errors of the estimators t_5, t_6 and t_7 proposed by Sharma and Singh (2014).The derivation of the correct expressions of the minimum mean squared errors of the estimators t_5, t_6 and t_7 proposed by Sharma and Singh (2014) are given in the following theorems.

Theorem 1.1. (a): The bias and *MSE* of the estimator t_5 to the first degree of approximation, are respectively given by

$$B(t_5) = (w_1 - 1)S_y^2, \tag{1.20}$$

$$MSE(t_5) = S_y^4[1 + w_1^2\{1 + f(\lambda_{40} - 1)\} + w_2^2 r^2 g^2 f(\lambda_{04} - 1) + 2w_1 w_2 grf(\lambda_{22} - 1) - 2w_1], \tag{1.21}$$

where $r = S_x^2 / S_y^2$ is the ratio of two variances.

Proof: To obtain the bias and *MSE* of t_5, we write

$$s_y^2 = S_y^2(1+e_0)\ ,\ s_x^2 = S_x^2(1+e_1)$$

such that

$$E(e_0) = E(e_1) = 0$$

and to the first degree of approximation,

$$E(e_0^2) = f(\lambda_{40}-1)\,, E(e_1^2) = f(\lambda_{04}-1) \text{ and } E(e_0 e_1) = f(\lambda_{22}-1)\,.$$

Expressing t_5 in terms of e's we have

$$\begin{aligned} t_5 &= w_1 S_y^2(1+e_0) + w_2\{S_x^2 - (1+g)S_x^2 + g(1+e_1)S_x^2\} \\ &= w_1 S_y^2(1+e_0) + w_2 g S_x^2 e_1 \end{aligned}$$

or

$$(t_5 - S_y^2) = w_1 S_y^2(1+e_0) + w_2 g S_x^2 e_1 - S_y^2$$

or

$$(t_5 - S_y^2) = S_y^2[w_1(1+e_0) + w_2 g r e_1 - 1]\,. \tag{1.22}$$

Taking expectation of both sides of (1.22) we get the bias of t_5 to the first degree of approximation as

$$B(t_5) = (w_1 - 1)S_y^2\,. \tag{1.23}$$

Squaring both sides of (1.22) we have

$$\begin{aligned} (t_5 - S_y^2)^2 &= S_y^4[1 + w_1^2(1 + 2e_0 + e_0^2) + w_2^2 g^2 r^2 e_1^2 \\ &+ 2w_1 w_2 g r(e_1 + e_0 e_1) - 2w_1(1+e_0) - 2w_2 g r e_1]\,. \end{aligned} \tag{1.24}$$

Taking expectation of both sides of (1.24) we get the *MSE* of t_5 to first degree of approximation as

$$MSE(t_5) = S_y^4[1 + w_1^2\{1 + f(\lambda_{40}-1)\} + w_2^2 g^2 r^2 f(\lambda_{04}-1) + 2w_1 w_2 g r f(\lambda_{22}-1) - 2w_1] \tag{1.25}$$

which proves the Theorem 1.1(a).

Theorem 1.1. (b): The optimum values of w_1 and w_2 that minimize the $MSE(t_5)$ at (1.25) are respectively given by

$$w_{10} = [1 + f(\lambda_{40}-1)(1-\rho^{*2})]^{-1}\,, \tag{1.26}$$

$$w_{20} = -\frac{C}{gr[1 + f(\lambda_{40}-1)(1-\rho^{*2})]}\,, \tag{1.27}$$

and the resulting minimum $MSE(t_5)$ is given by

$$\min.MSE(t_5) = \frac{S_y^4 f(\lambda_{40}-1)(1-\rho^{*2})}{[1+f(\lambda_{40}-1)(1-\rho^{*2})]}, \tag{1.28}$$

$$= \frac{\min.MSE(t_3)}{\left[1+\dfrac{\min.MSE(t_3)}{S_y^4}\right]}, \quad \text{[from (1.12)]}$$

Proof: Proof is simple so omitted.

Theorem 1.2. (a): The bias and *MSE* of the estimator t_6 to the first degree of approximation, are respectively given by

$$B(t_6) = S_y^2[k_1 + k_2 g^2 rf(\lambda_{04}-1)-1] \tag{1.29}$$

and

$$MSE(t_6) = S_y^4[1+k_1^2\{1+f(\lambda_{40}-1)\}+k_2^2 r^2 g^2 f(\lambda_{04}-1)$$
$$+2k_1k_2 grf(\lambda_{04}-1)(g+C)-2k_1-2k_2 g^2 rf(\lambda_{04}-1)]. \tag{1.30}$$

Proof: Expressing the estimator t_6 in terms of e's we have

$$t_6 = k_1 S_y^2(1+e_0)+k_2 S_x^2 g e_1[2-(1+g)+g(1+e_1)]$$
$$= k_1 S_y^2(1+e_0)+k_2 S_x^2 g e_1(1+g e_1)$$

or

$$(t_6 - S_y^2) = S_y^2[k_1(1+e_0)+k_2 gr(e_1+ge_1^2)-1]. \tag{1.31}$$

Taking the expectation of both sides of (1.31) we get the bias of t_6 to the first degree of approximation as

$$B(t_6) = S_y^2[k_1 + k_2 g^2 rf(\lambda_{04}-1)-1]. \tag{1.32}$$

Squaring both sides of (1.31) and neglecting terms of e's having power greater than two we have

$$(t_6 - S_y^2)^2 = S_y^4[1+k_1^2(1+2e_0+e_0^2)+k_2^2 g^2 r^2 e_1^2$$
$$+2k_1k_2 gr(e_1+e_0e_1+ge_1^2)-2k_1(1+e_0)-2k_2 gr(e_1+ge_1^2)]. \tag{1.33}$$

Taking expectation of both sides of (1.33) we get the *MSE* of t_6 to first degree of approximation as

$$MSE(t_6) = S_y^4[1 + k_1^2\{1 + f(\lambda_{40} - 1)\} + k_2^2 g^2 r^2 f(\lambda_{04} - 1)$$

$$+ 2k_1 k_2 grf(\lambda_{04} - 1)(g + C) - 2k_1 - 2k_2 g^2 rf(\lambda_{04} - 1)] \qquad (1.34)$$

which proves the Theorem1.2(a).

Theorem 1.2. (b): The optimum values of k_1 and k_2 that minimizes the $MSE(t_6)$ are respectively given by

$$k_{10} = \frac{[1 - gf(g + C)(\lambda_{04} - 1)]}{[1 + f\{(\lambda_{40} - 1) - (\lambda_{04} - 1)(g + C)^2]}, \qquad (1.35)$$

$$k_{20} = \frac{[gf(\lambda_{40} - 1) - C]}{gr[1 + f\{(\lambda_{40} - 1) - (\lambda_{04} - 1)(g + C)^2\}]} \qquad (1.36)$$

and the resulting minimum $MSE(t_6)$ is given by

$$\min.MSE(t_6) = \frac{S_y^4 f(\lambda_{40} - 1)[1 - g^2 f(\lambda_{04} - 1) - \rho^{*2}]}{[1 + f\{(\lambda_{40} - 1) - (\lambda_{04} - 1)(g + C)^2]}. \qquad (1.37)$$

Proof: Differentiating (1.34) partially with respect to k_1, k_2 and equating to zero we get the following equations:

$$k_1\{1 + f(\lambda_{40} - 1)\} + k_2 grf(\lambda_{04} - 1)(g + C) = 1, \qquad (1.38)$$

$$k_1(\lambda_{04} - 1)(g + C) + k_2 gr(\lambda_{04} - 1) = g(\lambda_{04} - 1). \qquad (1.39)$$

Solving (1.38) and (1.39) for k_1 and k_2, we get the optimum values of k_1 and k_2, respectively as given by (1.35) and (1.36).

Substituting the values of k_{10} and k_{20}, from (1.35) and (1.36) in (1.34) we get the resulting minimum *MSE* of t_6 given by (1.37).

This proves the Theorem 1.2(b).

Theorem 1.3. (a): The bias and *MSE* of the estimator t_7 to the first degree of approximation, are respectively given by

$$B(t_7) = S_y^2[m_1\{1 - gf(\lambda_{22} - 1)\} - 1], \qquad (1.40)$$

$$MSE(t_7) = S_y^4[1 + m_1^2\{1 + f[(\lambda_{40} - 1) + g(\lambda_{04} - 1)(g - 4C)]\}]$$
$$+ m_2^2 g^2 r^2 f(\lambda_{04} - 1) + 2m_1 m_2 grf(\lambda_{04} - 1)(C - g) - 2m_1\{1 - gf(\lambda_{22} - 1)\}]. \tag{1.41}$$

Proof: Expressing the estimator t_7 in terms of e's we have

$$t_7 = S_y^2(1 + e_0)[1 + g - g(1 + e_1)] + m_2[S_x^2 - (1 + g)S_x^2 + gS_x^2(1 + e_1)]$$
$$= S_y^2(1 + e_0)(1 - ge_1) + m_2 gS_x^2 e_1$$

or

$$(t_7 - S_y^2) = S_y^2[m_1(1 + e_0 - ge_1 - ge_0e_1) + m_2 gre_1 - 1]. \tag{1.42}$$

Taking expectation of both sides of (1.42) we get the bias of t_7 to the first degree of approximation as

$$B(t_7) = S_y^2[m_1\{1 - gf(\lambda_{22} - 1)\} - 1]. \tag{1.43}$$

Squaring both sides of (1.42) and neglecting terms of e's having power greater than two we have

$$(t_7 - S_y^2)^2 = S_y^4[1 + m_1^2(1 + 2e_0 - 2ge_1 + e_0^2 - 4ge_0e_1 + g^2e_1^2) + m_2^2 g^2 r^2 e_1^2$$
$$+ 2m_1 m_2 gr(e_1 + e_0e_1 - ge_1^2) - 2m_1(1 + e_0 - ge_1 - ge_0e_1) - 2m_2 gre_1]. \tag{1.44}$$

Taking expectation of both sides of (1.44) we get the *MSE* of t_7 to first degree of approximation as

$$MSE(t_7) = S_y^4[1 + m_1^2\{1 + f[(\lambda_{40} - 1)\} + g(\lambda_{04} - 1)(g - 4C)]\}$$
$$+ m_2^2 g^2 r^2 f(\lambda_{04} - 1) + 2m_1 m_2 grf(\lambda_{04} - 1)(C - g) - 2m_1\{1 - gf(\lambda_{22} - 1)\}].$$

This is same as given in (1.40). Thus the theorem is proved.

Theorem 1.3. (b): The optimum values of m_1 and m_2 that minimizes the $MSE(t_7)$ given by

$$m_{10} = \frac{[1 - gf(\lambda_{04} - 1)C]}{[1 + f\{(\lambda_{40} - 1) - (\lambda_{04} - 1)C(2g + C)\}]}, \tag{1.45}$$

$$m_{20} = -\frac{[1 - gf(\lambda_{04} - 1)C](C - g)}{gr[1 + f\{(\lambda_{40} - 1) - (\lambda_{04} - 1)C(2g + C)\}]} \tag{1.46}$$

and the resulting minimum $MSE(t_7)$ is given by

$$\min.MSE(t_7)=\frac{S_y^4 f(\lambda_{40}-1)[1-\rho^{*2}\{1+g^2 f(\lambda_{04}-1)\}]}{[1+f\{(\lambda_{40}-1)-(\lambda_{04}-1)C(2g+C)\}]}. \tag{1.47}$$

Proof: Differentiating (1.40) partially with resects to m_1, m_2 and equating to zero we get the following equations:

$$\begin{bmatrix}\{1+f[(\lambda_{40}-1)+g(\lambda_{04}-1)(g-4C)]\} & grf(\lambda_{04}-1)(C-g)\\ grf(\lambda_{04}-1)(C-g) & g^2r^2 f(\lambda_{04}-1)\end{bmatrix}\begin{bmatrix}m_1\\ m_2\end{bmatrix}=\begin{bmatrix}1-gf(\lambda_{04}-1)C\\ 0\end{bmatrix}. \tag{1.48}$$

Solving (1.48) we get the optimum values of m_1 and m_2 as given in (1.44) and (1.45) respectively. Substituting the optimum values of m_{10} and m_{20} of m_1 and m_2 respectively in the $MSE(t_7)$ at (1.41), we get the minimum *MSE* of t_7 as given by (1.47).

Thus the theorem is proved.

1.2. Efficiency comparison

This section compares some existing known estimators of the population variance S_y^2.

From (1.5), (1.8), (1.9), (1.12) and (1.16) we have

$$Var(t_0)-\underset{j=3,4}{\min.Min}(t_j)=f\,S_y^4(\lambda_{40}-1)\rho^{*2}\ \geq 0, \tag{1.49}$$

$$MSE(t_1)-\underset{j=3,4}{\min.Min}(t_j)=f\,S_y^4[\rho^*\sqrt{(\lambda_{40}-1)}-\sqrt{(\lambda_{04}-1)}]^2\ \geq 0, \tag{1.50}$$

$$MSE(t_2)-\underset{j=3,4}{\min.Min}(t_j)=f\,S_y^4[\rho^*\sqrt{(\lambda_{40}-1)}-g\sqrt{(\lambda_{04}-1)}]^2\ \geq 0. \tag{1.51}$$

It follows from (1.49) to (1.51) that the difference estimator t_3 [Das and Tripathi (1978)] and Shabbir (2006) estimator t_4 (at optimum condition) are better than the usual unbiased estimator s_y^2, Isaki's (1983) estimator t_1 and Upadhyaya and Singh's (1986) estimator t_2.

Now, we present the comparison of the estimators t_5, t_6 and t_7 due to Sharma and Singh (2014) with that of Das and Tripathi's (1978) difference estimator t_3 and Shabbir (2006) estimator t_4. From (1.12), (1.16), (1.28), (1.37) and (1.47) we have

$$\min.MSE(t_j)-\min.Min(t_5)=f\,S_y^4(\lambda_{40}-1)(1-\rho^{*2})\left[1-\frac{1}{\{1+f\,(\lambda_{40}-1)(1-\rho^{*2})\}}\right] \tag{1.52}$$

$$j=3,4 \qquad \geq 0,$$

$$\min.MSE(t_j)-\min.Min(t_6)=f\,S_y^4(\lambda_{40}-1)\left[(1-\rho^{*2})\left(1-\frac{1}{D}\right)+\frac{g^2 f(\lambda_{04}-1)}{D}\right] \tag{1.53}$$

$$j=3,4 \qquad \geq 0,$$

$$\min.MSE(t_j)-\min.Min(t_7)=f\,S_y^4(\lambda_{40}-1)\left[(1-\rho^{*2})\left(1-\frac{1}{D^*}\right)+\frac{\rho^{*2}g^2 f(\lambda_{04}-1)}{D^*}\right] \tag{1.54}$$

$$j=3,4 \qquad \geq 0,$$

where

$$D=[1+f\{(\lambda_{40}-1)-(\lambda_{04}-1)(g+C)^2\}] \quad \text{and}$$
$$D^*=[1+f\{(\lambda_{40}-1)-(\lambda_{04}-1)C(2g+C)\}].$$

It follows from (1.52) to (1.54) that the estimators t_5, t_6 and t_7 due to Sharma and Singh (2014) are better than Das and Tripathi's (1978) difference estimator t_3 and Shabbir (2006) estimator t_4 and hence better than the usual unbiased estimator s_y^2, Isaki's (1983) estimator t_1 and Upadhyaya and Singh's (1986) estimator t_2.

2. The suggested class of estimators for the population variance S_y^2

Keeping the form of Das and Tripathi's (1978) difference type estimator, Isaki's (1983) ratio-type estimator, Upadhyaya and Singh's (1986) estimators , Singh et al.'s (1988) estimator, Shabbir's (2006) estimator, Kadilar and Cingi's (2006, 2007) estimators, Shabbir and Gupta's (2007) estimator, Singh and

Solanki's (2013a, b) estimator, Solanki and Singh (2013) estimator, Singh et al.'s (2013, 2014) estimator, Sharma and Singh's (2014), Solanki et al. (2015) estimator and Singh and Pal (2016) estimators in view, we define a generalized class of estimators for S_y^2 as:

$$t_{SP}=\left[\phi_1 s_y^2\left\{\xi+(1-\xi)\left(\frac{s_{x(a,b)}^2}{S_x^2}\right)^{\alpha}\lambda\right\}+\phi_2(S_x^2-s_{x(a,b)}^2)\left\{\phi+(1-\phi)\left(\frac{s_{x(a,b)}^2}{S_x^2}\right)^{\delta}\right\}\right]$$

$$\times\left\{\theta+(1-\theta)\left(\frac{s_{x(a,b)}^2}{S_x^2}\right)^{p}\right\}\times\exp\left\{\frac{q(S_x^2-s_{x(a,b)}^2)}{(S_x^2+s_{x(a,b)}^2)}\right\}, \qquad (2.1)$$

where $s_{x(a,b)}^2=(aS_x^2+bs_x^2)/(a+b)$, (ϕ_1,ϕ_2) being suitable chosen constants, $(\phi,\xi,\theta,\lambda)$ are suitable chosen scalars such that $0\le(\phi,\xi,\lambda)\le 1$, λ may be equal to $\tau=(1+\psi C_{xy})/(1+\psi C_x^2)$ with $\psi=(1-f)/n$ and $f=n/N$; (α,δ,p,q) are scalars taking real values to generate ratio and product- type acceptable estimators; and (a,b) are either real numbers or the functions of the known parameters of the study variable *y* such as C_y coefficient of variation, (see Searls (1964), Lee (1981) and Singh (1986)), $\beta_2(y)\,(=\lambda_{40})$ (coefficient of kurtosis of the study variable *y* see Singh et al. (1973) and Searls and Intarapanich (1990)), coefficient of skewness $\beta_1(y)\,(=\lambda_{30}^2)$ of $y, \Delta(y)=(\beta_2(y)-\beta_1(y)-1)$ (see, Sen (1978), Upadhyaya and Singh (1984) and Singh and Agnihotri (2008)) or the functions of auxiliary variable *x* such as population mean $\bar{X}$, coefficient of variation C_x, coefficients of skewness $\beta_1(x)\,(=\lambda_{03}^2)$ and kurtosis $\beta_2(x)\,(=\lambda_{04})$ and the parameter $\Delta(x)=(\beta_2(x)-\beta_1(x)-1)$ or the population correlation coefficient ρ between the study variable *y* and the auxiliary variable *x*.

We would like to remark that for various values of the parameters in (2.1), we get some existing known estimators as shown in Table 2.1. Many other estimators can also be generated from the proposed family of estimators t_{SP} for suitable values of scalars (ϕ_1, ξ, α, λ, a, b, ϕ_2, ϕ, δ, θ, p, q).

Table 2.1. Some known members of proposed class of estimators

Values of the constants (λ, ϕ_1, ϕ_2, θ, ξ, ϕ, α, δ, p, q, a, b)	**Estimator**
(1, 1, 0, -, -, -, 0, -, 0, 0, -, -)	$t_{SP(1)}=s_y^2$
(-, 1, d, -, 1, -, -, 0, 0, 0, 0, -)	$t_{SP(2)}=s_y^2+d(S_x^2-s_x^2)$ Das and Tripathi's (1978) estimator

(1, 1, 0, -, 0, -, -α, -, 0, 0, 0, -)	$t_{SP(3)} = s_y^2\left(\frac{S_x^2}{s_x^2}\right)^{\alpha}$ Das and Tripathi's (1978) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, 1-b, b)	$t_{SP(4)} = s_y^2\left(\frac{S_x^2}{s_x^2 + b(s_x^2 - S_x^2)}\right)$ Das and Tripathi's (1978) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, 0, -)	$t_{SP(5)} = s_y^2(S_x^2 / s_x^2)$ Isaki's (1983) ratio estimator
(1, 1, 0, -, 0, -,1, -, 0, 0, $(1+g)$, $-g$)	$t_{SP(6)} = s_y^2(s_x^{*2} / S_x^2)$ Upadhyaya and Singh's (1986) estimator
(1, ϕ_1, ϕ_2, -, -, -, 0, 0, 0, 0, 0, -)	$t_{SP(7)} = \phi_1 s_y^2 + \phi_2(S_x^2 - s_x^2)$ Singh et al.'s(1988) estimator
(1, ϕ_1, 0, -, 0, -, -1, -, 0, 0, 0, -)	$t_{SP(8)} = \phi_1 s_y^2(S_x^2 / s_x^2)$ Prasad and Singh's (1990) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, $\frac{\beta_2(x)}{S_x^2}$, 1)	$t_{SP(9)} = s_y^2\left(\frac{S_x^2 + \beta_2(x)}{s_x^2 + \beta_2(x)}\right)$ Upadhyaya and Singh's (1999) estimator
(1, 1, 0, -, A, -, -1, -, 0, 0, $\frac{\beta_2(x)}{S_x^2}$, 1)	$t_{SP(10)} = As_y^2 + (1-A)s_y^2\left(\frac{S_x^2 + \beta_2(x)}{s_x^2 + \beta_2(x)}\right)$ Chandra and Singh's(2001) estimator
(1, 1, 0, -, $1+w$, -,1, -, 0, 0, $\frac{\beta_2(x)}{S_x^2}$, 1)	$t_{SP(11)} = (1+w)s_y^2 - ws_y^2\left(\frac{s_x^2 + \beta_2(x)}{S_x^2 + \beta_2(x)}\right)$ Chandra and Singh's(2001) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, $-\frac{C_x}{S_x^2}$, 1)	$t_{SP(12)} = s_y^2\left(\frac{S_x^2 - C_x}{s_x^2 - C_x}\right)$ Kadilar and Cingi's (2005) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, $-\frac{\beta_2(x)}{S_x^2}$,1)	$t_{SP(13)} = s_y^2\left(\frac{S_x^2 - \beta_2(x)}{s_x^2 - \beta_2(x)}\right)$ Kadilar and Cingi's (2005) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, $-\frac{C_x}{S_x^2}$, $\beta_2(x)$)	$t_{SP(14)} = s_y^2\left(\frac{S_x^2\beta_2(x) - C_x}{s_x^2\beta_2(x) - C_x}\right)$ Kadilar and Cingi's (2005) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, $-\frac{\beta_2(x)}{S_x^2}$, C_x)	$t_{SP(15)} = s_y^2\left(\frac{S_x^2 C_x - \beta_2(x)}{s_x^2 C_x - \beta_2(x)}\right)$ Kadilar and Cingi's (2005) estimator
(1, 1, 0, -, ξ, -1, -, 0, 0, $(1+g)$, $-g$)	$t_{SP(16)} = \xi s_y^2 + (1-\xi)s_y^2\left(\frac{s_x^{*2}}{S_x^2}\right)$ Shabbir's (2006) estimator
(τ , 1, 0, -, ξ, -, -1, -, 0, 0, 0, -)	$t_{SP(17)} = \xi s_y^2 + (1-\xi)s_y^2\left(\frac{S_x^2}{s_x^2}\right)\tau$ Kadilar and Cingi's (2006) estimators

(1, ϕ_1, ϕ_2, -, -, -, 0, 0, 0, 1, 0, -)	$t_{SP(18)} = [\phi_1 s_y^2 + \phi_2 (S_x^2 - s_x^2)] \exp\left\{\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right\}$ Shabbir and Gupta's (2007) estimator
(1, ϕ_1, ϕ_2, 2, -, -, 0, 0, p, 0, 0, -)	$t_{SP(19)} = [\phi_1 s_y^2 + \phi_2 (S_x^2 - s_x^2)]\left[2 - \left(\frac{s_x^2}{S_x^2}\right)^p\right]$ Gupta and Shabbir's (2008) estimator
(1, ϕ_1, ϕ_2, 2, -, -, 0, 0, 1, 0, 0, -)	$t_{SP(20)} = [\phi_1 s_y^2 + \phi_2 (S_x^2 - s_x^2)]\left[2 - \left(\frac{s_x^2}{S_x^2}\right)\right]$ Gupta and Shabbir's (2008) estimator
(1, ϕ_1, ϕ_2, 2, -, -, 0, 0, -1, 0, 0, -)	$t_{SP(21)} = [\phi_1 s_y^2 + \phi_2 (S_x^2 - s_x^2)]\left[2 - \left(\frac{S_x^2}{s_x^2}\right)\right]$ Gupta and Shabbir's (2008) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, $\frac{\eta(1-\alpha^*)S_x^2 - \upsilon)}{S_x^2}$, $\alpha^*\eta$)	$t_{SP(22)} = s_y^2 \frac{(\eta S_x^2 - \upsilon)}{\{\alpha^*(\eta s_x^2 - \upsilon) + (1-\alpha^*)(\eta S_x^2 - \upsilon)\}}$ Yadav and Pandey's (2012) type estimator and Singh and Malik's (2014) type estimator
(1, k, 0, -, 0, -, -1, -, 0, 0, $\frac{\eta(1-\alpha^*)S_x^2 - \upsilon)}{S_x^2}$, $\alpha^*\eta$)	$t_{SP(23)} = k s_y^2 \frac{(\eta S_x^2 - \upsilon)}{\{\alpha^*(\eta s_x^2 - \upsilon) + (1-\alpha^*)(\eta S_x^2 - \upsilon)\}}$ Yadav and Pandey 's(2012) type estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, $\frac{Q_1}{S_x^2}$, 1)	$t_{SP(24)} = s_y^2 \left(\frac{S_x^2 + Q_1}{s_x^2 + Q_1}\right)$ Subramani and Kumarapandiyan's (2012b) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, $\frac{Q_3}{S_x^2}$, 1)	$t_{SP(25)} = s_y^2 \left(\frac{S_x^2 + Q_3}{s_x^2 + Q_3}\right)$ Subramani and Kumarapandiyan's (2012b) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, $\frac{Q_2}{S_x^2}$, 1)	$t_{SP(26)} = s_y^2 \left(\frac{S_x^2 + Q_2}{s_x^2 + Q_2}\right)$ Subramani and Kumarapandiyan's (2012b) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, $\frac{Q_d}{S_x^2}$, 1)	$t_{SP(27)} = s_y^2 \left(\frac{S_x^2 + Q_d}{s_x^2 + Q_d}\right)$ Subramani and Kumarapandiyan's (2012b) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, $\frac{Q_a}{S_x^2}$, 1)	$t_{SP(28)} = s_y^2 \left(\frac{S_x^2 + Q_a}{s_x^2 + Q_a}\right)$ Subramani and Kumarapandiyan's (2012b) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, $\frac{M_d}{S_x^2}$, 1)	$t_{SP(29)} = s_y^2 \left(\frac{S_x^2 + M_d}{s_x^2 + M_d}\right)$ Subramani and Kumarapandiyan's (2012a) estimator

(1, $w_1, w_2 \dfrac{(a+b)}{b}$, 0, -, -, 0, 0, -1, 0, $\dfrac{h}{S_x^2}$, b)	$t_{SP(30)} = [w_1 s_y^2 + w_2 (S_x^2 - s_x^2)]\left(\dfrac{bS_x^2 + h}{bs_x^2 + h}\right)$ Singh and Solanki's (2013a) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, $\dfrac{C_x}{S_x^2}$, 1)	$t_{SP(31)} = s_y^2\left(\dfrac{S_x^2 + C_x}{s_x^2 + C_x}\right)$ Singh and Solanki's (2013a) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, $\dfrac{C_x}{S_x^2}$, $\beta_2(x)$)	$t_{SP(32)} = s_y^2\left(\dfrac{S_x^2\beta_2(x) + C_x}{s_x^2\beta_2(x) + C_x}\right)$ Singh and Solanki's (2013a) estimator
(1, 1, 0, -, 0 -, -1, -, 0, 0, $\dfrac{\beta_2(x)}{S_x^2}$, C_x)	$t_{SP(33)} = s_y^2\left(\dfrac{S_x^2 C_x + \beta_2(x)}{s_x^2 C_x + \beta_2(x)}\right)$ Singh and Solanki's (2013a) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, $\dfrac{Q_3}{S_x^2}$, ρ)	$t_{SP(34)} = s_y^2\left(\dfrac{S_x^2\rho + Q_3}{s_x^2\rho + Q_3}\right)$ Khan and Shabbir's (2013) estimator
(1, 1, 0, -, 0, -, -1, -, 0, 0, $\dfrac{\gamma L_i^2}{S_x^2}$, 1) γ being a constant such that $0 \le \gamma \le 1$	$t_{SP(35)} = s_y^2\left(\dfrac{S_x^2 + \gamma L_i^2}{s_x^2 + \gamma L_i^2}\right)$ i= 1 to 6, $L_1 = Q_1$, $L_2 = Q_2$, $L_3 = Q_3$, $L_4 = Q_r$, $L_5 = Q_d$, $L_6 = Q_a$, where Q_1 (first quartile), Q_2 (second range), Q_3 (third quartile), $Q_d = (Q_3 - Q_1)/2$, $Q_a = (Q_3 + Q_1)/2$. Singh et al.'s (2013) estimator
(1, w_1, w_2, -, -, -, 0, 0, 0, 0, $(1+g)$, $-g$)	$t_{SP(36)} = w_1 s_y^2 + w_2 (S_x^2 - s_x^{*2})$ Sharma and Singh's (2014) estimator
(1, k_1, k_2, -, -, 2, 0, 1, 0, 0, $(1+g)$, $-g$)	$t_{SP(37)} = k_1 s_y^2 + k_2 (S_x^2 - s_x^{*2})[2 - (s_x^{*2}/S_x^2)]$ Sharma and Singh's (2014) estimator
(1, m_1, m_2, -, 0, -, 1, 0, 0, 0, $(1+g)$, $-g$)	$t_{SP(38)} = m_1 s_y^2 (s_x^{*2}/S_x^2) + m_2 (S_x^2 - s_x^{*2})$ Sharma and Singh's (2014) estimator
(1, w_1, $w_2 \dfrac{(a+b)}{b}$, 0, -, -, 0, 0, -1, 0, $\dfrac{\eta^* L^2}{S_x^2}$, δ^*)	$t_{SP(39)} = [w_1 s_y^2 + w_2 (S_x^2 - s_x^2)]\left(\dfrac{\delta^* S_x^2 + \eta^* L^2}{\delta^* s_x^2 + \eta^* L^2}\right)$ Singh and Pal 's (2016) estimator
(1, w_1, $w_2 \dfrac{(a+b)}{b}$, -, -, -, 0, 0, 0, 1, $\dfrac{\eta^* L^2}{S_x^2}$, δ^*)	$t_{SP(40)} = [w_1 s_y^2 + w_2 (S_x^2 - s_x^2)]\exp\left(\dfrac{\delta^*(S_x^2 - s_x^2)}{\delta^*(S_x^2 + s_x^2) + 2\eta^* L^2}\right)$ Singh and Pal's (2016) estimator

where $(A, w, \alpha^*, h, w_1, w_2, k_1, k_2)$ are suitable chosen constants and (δ^*, L) are either real constants or function of known parameter of an auxiliary variable x and η^* being a constant such that $|\eta^*| \le 1$.

2.1. Derivation of the expressions of bias and mean squared error (*MSE*) of the class of estimators t_{SP}

To obtain the bias and *MSE* of the class of estimators t_{SP} at (2.1) in terms of e's we have

$$t_{SP} = [\phi_1 S_y^2(1+e_0)\{\xi+(1-\xi)(1+b^*e_1)^\alpha \lambda\} - \phi_2 b^* S_x^2 e_1\{\phi+(1-\phi)(1+b^*e_1)^\delta\}]$$

$$\times\{\theta+(1-\theta)(1+b^*e_1)^p\}\times\exp\left\{-\left(\frac{b^*q}{2}\right)e_1\left(1+\frac{b^*}{2}e_1\right)^{-1}\right\}, \qquad (2.2)$$

where $b^* = b/(a+b)$.

We assume that $|b^*e_1|<1$ so that $(1+b^*e_1)^\alpha$, $(1+b^*e_1)^\delta$, $(1+b^*e_1)^p$ and $\left(1+\frac{b^*}{2}e_1\right)^{-1}$ are expandable. Now expanding the right hand side of (2.2), we have

$$t_{SP} = \left[\phi_1 S_y^2(1+e_0)\left\{\xi+(1-\xi)\lambda\left[1+\alpha b^* e_1+\frac{\alpha(\alpha-1)}{2}b^{*2}e_1^2+...\right]\right\}\right.$$

$$\left.-\phi_2 b^* S_x^2 e_1\left\{\phi+(1-\phi)\left[1+\delta b^* e_1+\frac{\delta(\delta-1)}{2}b^{*2}e_1^2+...\right]\right\}\right]$$

$$\times\left\{\theta+(1-\theta)\left[1+pb^*e_1+\frac{p(p-1)}{2}b^{*2}e_1^2+...\right]\right\}$$

$$\times\left\{1-\frac{b^*q}{2}e_1\left(1+\frac{b^*}{2}e_1\right)^{-1}+\frac{b^{*2}q^2}{8}e_1^2\left(1+\frac{b^*}{2}e_1\right)^{-2}-...\right\}$$

$$= S_y^2\left[\phi_1(1+e_0)\left\{\xi_0+(1-\xi)\lambda\alpha b^*\left(e_1+\frac{(\alpha-1)b^*}{2}e_1^2+...\right)\right\}\right.$$

$$\left.-\phi_2 b^* r e_1\left\{1+(1-\phi)\delta b^*\left(e_1+\frac{(\delta-1)b^*}{2}e_1^2+...\right)\right\}\right]\times\left\{1+(1-\theta)pb^*\left(e_1+\frac{(p-1)b^*}{2}e_1^2+...\right)\right\}$$

$$\times\left\{1-\frac{b^*q}{2}e_1+\frac{b^{*2}q(q+2)}{8}e_1^2-...\right\}$$

$$= S_y^2\left[\phi_1\left\{\xi_0(1+e_0)+(1-\xi)\lambda\alpha b^*(e_1+e_0e_1)+\frac{(1-\xi)\lambda\alpha(\alpha-1)b^{*2}}{2}e_1^2+...\right\}\right.$$

$$-\phi_2 b^* r\{e_1+(1-\phi)\delta b^* e_1^2+...\}]\times\left\{1+(1-\theta)pb^* e_1+(1-\theta)\frac{p(p-1)b^{*2}}{2}e_1^2\right.$$

$$\left.-\left(\frac{b^* q}{2}\right)e_1+\left(\frac{b^{*2}pq(1-\theta)}{2}\right)e_1^2+\frac{b^{*2}q(q+2)}{8}e_1^2+...\right\}$$

$$=S_y^2\left[\phi_1\left\{\xi_0(1+e_0)+(1-\xi)\lambda\alpha b^*(e_1+e_0e_1)+\frac{(1-\xi)\lambda\alpha(\alpha-1)}{2}b^{*2}e_1^2+...\right\}\right.$$

$$\left.-\phi_2 b^* r\{e_1+(1-\phi)\delta b^* e_1^2+...\}\right]\times\left\{1+u_1 b^* e_1+\frac{b^{*2}u_2}{2}e_1^2+...\right\}$$

$$=S_y^2\left[\phi_1\left\{\xi_0(1+e_0)+(1-\xi)\lambda\alpha b^*(e_1+e_0e_1)+\frac{(1-\xi)\lambda\alpha(\alpha-1)}{2}b^{*2}e_1^2+\xi_0 b^* u_1(e_1+e_0e_1)\right.\right.$$

$$\left.\left.+(1-\xi)\lambda\alpha b^{*2}u_1(e_1^2+e_0e_1^2)+\frac{b^{*2}u_2}{2}\xi_0(e_1^2+e_0e_1^2)+...\right\}-\phi_2 b^* r\{e_1+(1-\phi)\delta b^* e_1^2+b^* u_1 e_1^2+....\}\right]$$

$$=S_y^2\left[\phi_1\left\{\xi_0+\xi_0e_0+b^*\xi^* e_1+b^*\xi^* e_0e_1+\left(\frac{b^{*2}\theta^*}{2}\right)e_1^2+b^{*2}\left((1-\xi)\lambda\alpha u_1+\frac{u_2\xi_0}{2}\right)e_0e_1^2+...\right\}\right.$$

$$\left.-\phi_2 b^* r\{e_1+b^*[(1-\phi)\delta+u_1]e_1^2+....\}\right], \tag{2.3}$$

where

$$\xi_0=[\xi+(1-\xi)\lambda],\ \ \xi^*=[\xi u_1+(1-\xi)\lambda(\alpha+u_1)],\ \ u_1=\left[(1-\theta)p-\frac{q}{2}\right],$$

$$\theta^*=[\alpha(\alpha-1)(1-\xi)\lambda+\xi_0u_2)],\ \ u_2=\left[(1-\theta)p(p-q-1)+\frac{q(q+2)}{4}\right],$$

$\phi^*=[(1-\phi)\delta+u_1]=[(1-\phi)\delta+(1-\theta)p-(q/2)],\ r=S_x^2/S_y^2,$

$b^*=b/(a+b)$.

Neglecting terms of e's in (2.3) having power greater than two, we have

$$t_{SP}\cong S_y^2\left[\phi_1\left\{\xi_0+\xi_0e_0+b^*\xi^* e_1+b^*\xi^* e_0e_1+\left(\frac{b^{*2}\theta^*}{2}\right)e_1^2\right\}-\phi_2 b^* r\{e_1+b^*\phi^* e_1^2\}\right]$$

or

$$(t_{SP}-S_y^2)\equiv S_y^2\left[\phi_1\left\{\xi_0+\xi_0e_0+b^*\xi^* e_1+b^*\xi^* e_0e_1+\left(\frac{b^{*2}\theta^*}{2}\right)e_1^2\right\}-\phi_2 b^* r\{e_1+b^*\phi^* e_1^2\}-1\right], \tag{2.4}$$

Taking expectation of both sides of (2.4) we get the bias of t_{SP} to the first degree of approximation as

$$B(t_{SP}) = S_y^2\left[\phi_1\left\{\xi_0 + f(\lambda_{04}-1)b^*\left(\frac{b^*\theta^*}{2}+\xi^*C\right)\right\} - \phi_2 r b^{*2}\phi^* f(\lambda_{04}-1) - 1\right]. \tag{2.5}$$

Squaring both sides of (2.4) and neglecting terms of e's having power greater than two, we have

$$\begin{aligned}(t_{SP}-S_y^2)^2 &= S_y^4[1+\phi_1^2\{\xi_0^2(1+2e_0+e_0^2)+2b^*\xi_0\xi^*(e_1+2e_0e_1)+b^{*2}(\xi^{*2}+\theta^*\xi_0)e_1^2\}\\ &+\phi_2^2 r^2 b^{*2}e_1^2 - 2\phi_1\phi_2 b^* r\{\xi_0(e_1+e_0e_1)+b^*(\xi^*+\phi^*\xi_0)e_1^2\} - 2\phi_1\{\xi_0(1+e_0)\\ &+b^*\xi^*(e_1+e_0e_1)+((b^{*2}\theta^*)/2)e_1^2\}+2\phi_2 b^* r(e_1+b^*\phi^* e_1^2)].\end{aligned} \tag{2.6}$$

Taking expectation of both sides of (2.6) we get the *MSE* of t_{SP} to the first degree of approximation as

$$MSE(t_{SP}) = S_y^4[1+\phi_1^2 A_1+\phi_2^2 A_2 - 2\phi_1\phi_2 A_3 - 2\phi_1 A_4 + 2\phi_2 A_5], \tag{2.7}$$

where

$$A_1 = [\xi_0^2 + f\{\xi_0^2(\lambda_{40}-1)+b^*(\lambda_{04}-1)[b^*(\xi^{*2}+\theta^*\xi_0)+4\xi_0\xi^*C]\}],$$

$$A_2 = r^2 b^{*2} f(\lambda_{04}-1),$$

$$A_3 = b^* r f(\lambda_{04}-1)[b^*(\xi^*+\phi^*\xi_0)+\xi_0 C],$$

$$A_4 = [\xi_0 + b^* f(\lambda_{04}-1)((b^*\theta^*/2)+\xi^* C)],$$

$$A_5 = b^{*2}\phi^* r f(\lambda_{04}-1),$$

$$f = ((1/n)-(1/N)).$$

Differentiating (2.7) partially with respect to ϕ_1 and ϕ_2 and equating to zero, we have

$$\begin{bmatrix} A_1 & -A_3 \\ -A_3 & A_2 \end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{bmatrix} A_4 \\ -A_5\end{bmatrix}. \tag{2.8}$$

Solving (2.8) we get the optimum values of ϕ_1 and ϕ_2 as

$$\left.\begin{aligned}\phi_1 &= \frac{(A_2A_4 - A_3A_5)}{(A_1A_2 - A_3^2)} = \phi_{10}\,(say)\\ \phi_2 &= \frac{(A_3A_4 - A_1A_5)}{(A_1A_2 - A_3^2)} = \phi_{20}\,(say)\end{aligned}\right\}. \tag{2.9}$$

Putting (2.9) in (2.8) we get the resulting minimum MSE of t_{SP} as

$$\min.MSE(t_{SP}) = S_y^4\left[1-\frac{(A_2A_4^2-2A_3A_4A_5+A_1A_5^2)}{(A_1A_2-A_3^2)}\right]. \tag{2.10}$$

Thus we established the following theorem.

Theorem 2.1: Up to the first degree of approximation,

$$MSE(t_{SP}) \geq S_y^4\left[1-\frac{(A_2A_4^2-2A_3A_4A_5+A_1A_5^2)}{(A_1A_2-A_3^2)}\right]$$

with equality holding if

$$\phi_1 = \phi_{10},$$
$$\phi_2 = \phi_{20},$$

where ϕ_{i0}'s (i=1, 2)are defined in (2.9).

2.2. Special Case-I $(\lambda = 1)$

Putting $\lambda = 1$ in (2.1) we get the class of estimators of S_y^2 as

$$t_{SP}^{(1)} = \left[\phi_1 s_y^2\left\{\xi+(1-\xi)\left(\frac{s_{x(a,b)}^2}{S_x^2}\right)^{\alpha}\right\}+\phi_2(S_x^2-s_{x(a,b)}^2)\left\{\phi+(1-\phi)\left(\frac{s_{x(a,b)}^2}{S_x^2}\right)^{\delta}\right\}\right]$$
$$\times\left\{\theta+(1-\theta)\left(\frac{s_{x(a,b)}^2}{S_x^2}\right)^{p}\right\}\times\exp\left\{\frac{q(S_x^2-s_{x(a,b)}^2)}{(S_x^2+s_{x(a,b)}^2)}\right\}. \tag{2.11}$$

Substitution of $\lambda = 1$ in (2.5) and (2.7) yields the bias and *MSE* of the class of estimators to the first degree of approximation, respectively as

$$B(t_{SP}^{(1)}) = -S_y^2\left[1-\phi_1\left\{1+f(\lambda_{04}-1)b^*\left(\frac{b^*\theta_1^*}{2}+\xi_1^*C\right)\right\}+\phi_2 r b^{*2}\phi^* f(\lambda_{04}-1)\right] \tag{2.12}$$

and

$$MSE(t_{SP}^{(1)}) = S_y^4[1+\phi_1^2A_1^*+\phi_2^2A_2-2\phi_1\phi_2A_3^*-2\phi_1A_4^*+2\phi_2A_5], \tag{2.13}$$

where

$$A_1^* = [1+f\{(\lambda_{40}-1)+b^*(\lambda_{04}-1)[b^*(\xi_1^{*2}+\theta_1^*)+4\xi_1^*C]\}],$$

$$A_3^* = b^* rf(\lambda_{04} - 1)[b^*(\xi_1^* + \phi^*) + C],$$
$$A_4^* = [1 + b^* f(\lambda_{04} - 1)((b^*\theta_1^*/2) + \xi_1^* C)],$$
$$\xi_0 = 1, \xi_1^* = [u_1 + \alpha(1-\xi)], \theta_1^* = [\alpha(\alpha-1)(1-\xi) + u_2].$$

The $MSE(t_{SP}^{(1)})$ is minimized for

$$\left.\begin{aligned} \phi_1 &= \frac{(A_2 A_4^* - A_3^* A_5)}{(A_1^* A_2 - A_3^{*2})} = \phi_{10}^{(1)} \\ \phi_2 &= \frac{(A_3^* A_4^* - A_1^* A_5)}{(A_1^* A_2 - A_3^{*2})} = \phi_{20}^{(1)} \end{aligned}\right\}. \tag{2.14}$$

Thus the resulting minimum $MSE(t_{SP}^{(1)})$ is given by

$$\min.MSE(t_{SP}^{(1)}) = S_y^4\left[1 - \frac{(A_2 A_4^{*2} - 2A_3^* A_4^* A_5 + A_1^* A_5^2)}{(A_1^* A_2 - A_3^{*2})}\right]. \tag{2.15}$$

Thus we established the following corollary.

Corollary 2.1: Up to the first degree of approximation,

$$MSE(t_{SP}^{(1)}) \geq S_y^4\left[1 - \frac{(A_2 A_4^{*2} - 2A_3^* A_4^* A_5 + A_1^* A_5^2)}{(A_1^* A_2 - A_3^{*2})}\right]$$

with equation holding if

$$\phi_1 = \phi_{10}^{(1)},$$
$$\phi_2 = \phi_{20}^{(1)}.$$

From (2.10) and (2.15) we have

$$\min.MSE(t_{SP}^{(1)}) - \min.MSE(t_{SP})$$
$$= S_y^4\left[\frac{(A_2 A_4^2 - 2A_3 A_4 A_5 + A_1 A_5^2)}{(A_1 A_2 - A_3^2)} - \frac{(A_2 A_4^{*2} - 2A_3^* A_4^* A_5 + A_1^* A_5^2)}{(A_1^* A_2 - A_3^{*2})}\right]$$

which is positive if

$$\frac{(A_2 A_4^2 - 2A_3 A_4 A_5 + A_1 A_5^2)}{(A_1 A_2 - A_3^2)} > \frac{(A_2 A_4^{*2} - 2A_3^* A_4^* A_5 + A_1^* A_5^2)}{(A_1^* A_2 - A_3^{*2})}. \tag{2.16}$$

Thus the proposed family of estimators t_{SP} would be more efficient than the family of estimators $t_{SP}^{(1)}$ as long as inequality (2.16) is satisfied.

2.3. Special Case-II $(\phi_1, \lambda) = (1,1)$

Putting $(\phi_1, \lambda) = (1,1)$ in (2.1) we get class of estimators of S_y^2 as

$$t_{SP}^{(2)} = \left[s_y^2 \left\{ \xi + (1-\xi)\left(\frac{s_{x(a,b)}^2}{S_x^2} \right)^{\alpha} \right\} + \phi_2 (S_x^2 - s_{x(a,b)}^2) \left\{ \phi + (1-\phi)\left(\frac{s_{x(a,b)}^2}{S_x^2} \right)^{\delta} \right\} \right]$$

$$\times \left\{ \theta + (1-\theta)\left(\frac{s_{x(a,b)}^2}{S_x^2} \right)^{p} \right\} \times \exp\left\{ \frac{q(S_x^2 - s_{x(a,b)}^2)}{(S_x^2 + s_{x(a,b)}^2)} \right\}. \tag{2.17}$$

Inserting $(\phi_1, \lambda) = (1,1)$ in (2.5) and (2.7) yield the bias and *MSE* of $t_{SP}^{(2)}$ to the first degree of approximation, respectively given by

$$B(t_{SP}^{(2)}) = S_y^2 b^* f(\lambda_{04} - 1)\left[\left(\frac{b^* \theta_1^*}{2} \right) + \xi_1^* C - \phi_2 r b^* \phi^* \right] \tag{2.18}$$

and

$$MSE(t_{SP}^{(2)}) = S_y^4 [1 + A_1^* - 2A_4^* + \phi_2^2 A_2 - 2\phi_2 (A_3^* - A_5)]. \tag{2.19}$$

The $MSE(t_{SP}^{(2)})$ is minimized for

$$\phi_2 = \frac{(A_3^* - A_5)}{A_2} = \phi_{20}^* \, (say). \tag{2.20}$$

Thus the resulting minimum $MSE(t_{SP}^{(2)})$ is given by

$$\min .MSE(t_{SP}^{(2)}) = S_y^4 \left[1 - A_1^* - 2A_4^* - \frac{(A_3^* - A_5)^2}{A_2} \right], \tag{2.21}$$

$$= f S_y^4 (\lambda_{40} - 1)(1 - \rho^{*2})$$

which equals to the minimum *MSE* of the difference estimator t_3 defined in (1.3).

Now, we state the following corollary.

Corollary 2.2: Up to the first degree of approximation,

$$MSE(t_{SP}^{(2)}) \geq f S_y^4 (\lambda_{40} - 1)(1 - \rho^{*2})$$

with equality holding if

$$\phi_2 = \phi_{20}^* .$$

From (2.15) and (2.21) we have

$$\min.MSE(t_{SP}^{(2)})-\min.MSE(t_{SP}^{(1)})=\frac{S_y^4[A_3^*(A_3^*-A_5)-A_2(A_1^*-A_4^*)]^2}{A_2(A_1^*A_2-A_3^{*2})} \tag{2.22}$$

which is always positive. It follows that the proposed family of estimators $t_{SP}^{(1)}$ is better than the family of estimators $t_{SP}^{(2)}$ and the difference type estimators t_3 in (1.3) at their optimum conditions.

2.4. Special Case-III $(\phi_1=1)$

For $\phi_1=1$, the suggested class of estimators t_{SP} reduces to the class of estimators

$$t_{SP}^{(3)}=\left[s_y^2\left\{\xi+(1-\xi)\left(\frac{s_{x(a,b)}^2}{S_x^2}\right)^{\alpha}\lambda\right\}+\phi_2(S_x^2-s_{x(a,b)}^2)\left\{\phi+(1-\phi)\left(\frac{s_{x(a,b)}^2}{S_x^2}\right)^{\delta}\right\}\right]$$
$$\times\left\{\theta+(1-\theta)\left(\frac{s_{x(a,b)}^2}{S_x^2}\right)^{p}\right\}\times\exp\left\{\frac{q(S_x^2-s_{x(a,b)}^2)}{(S_x^2+s_{x(a,b)}^2)}\right\}. \tag{2.23}$$

Putting $\phi_1=1$ in (2.5) and (2.7) we get the bias and *MSE* of the estimator $t_{SP}^{(3)}$ to the first degree of approximation, respectively given by

$$B(t_{SP}^{(3)})=-S_y^2\left[1-\left\{\xi_0+fb^*(\lambda_{04}-1)\left(\frac{b^*\theta^*}{2}+\xi^*C\right)\right\}+\phi_2 rb^{*2}\phi^* f(\lambda_{04}-1)\right] \tag{2.24}$$

and

$$MSE(t_{SP}^{(3)})=S_y^4[1+A_1-2A_4+\phi_2^2A_2-2\phi_2(A_3-A_5)]. \tag{2.25}$$

The $MSE(t_{SP}^{(3)})$ is minimized when

$$\phi_2=\frac{(A_3-A_5)}{A_2}=\phi_{20}^{*(1)}(say). \tag{2.26}$$

Thus the resulting minimum $MSE(t_{SP}^{(3)})$ is given by

$$\min.MSE(t_{SP}^{(3)})=S_y^4\left[1+A_1-2A_2-\frac{(A_3-A_5)^2}{A_2}\right]. \tag{2.27}$$

Now, we state the following corollary.

Corollary 2.3. To the first degree of approximation,

$$MSE(t_{SP}^{(3)}) \geq S_y^4\left[1 + A_1 - 2A_2 - \frac{(A_3 - A_5)^2}{A_2}\right]$$

with equality holding if

$$\phi_2 = \phi_{20}^{*(1)}.$$

From (2.10) and (2.27) we have

$$\min.MSE(t_{SP}^{(3)}) - \min.MSE(t_{SP}) = \frac{S_y^4[A_2(A_1 - A_4) - A_3(A_3 - A_5)]^2}{A_2(A_1A_2 - A_3^2)}$$

$$\geq 0 \qquad (2.28)$$

which clearly indicates that the t_{SP} family of estimators is more efficient than that of the $t_{SP}^{(3)}$ family of estimators .

Concluding remarks

This paper intends to suggest a new family of estimators for the variance S_y^2 of the variable *y* of interest when the population variance S_x^2 of the auxiliary variable *x* is known. The proposed family generalizes that of the several estimators ($t_{SP(i)}$, i= 1 to 38) as listed in Table 2.1. We have obtained the bias and mean squared error (*MSE*) expressions up to first order of approximation in simple random sampling without replacement (*SRSWOR*). From the bias and *MSE* expressions of the suggested family, one can easily derive the bias and *MSE* expressions of existing known estimators as well as those of potential new proposals. The present study unifies several results at one place.

The family is certainly not exhaustive but it can act as different against the proliferation of equivalent proposals that could be appearing in the future. Three subclasses of the proposed family are identified and their properties are studied. We have also given the comparisons among the proposed class of estimators and the three subclasses of estimators. It has been theoretically shown that the proposed class of estimators is more efficient than the difference type estimator t_3 due to Das and Tripathi (1978) and hence the usual unbiased estimator s_y^2 and Isaki (1983) ratio estimator t_1 and several other estimators. This paper also provides the correct *MSE* expressions of the estimators (t_5, t_6, t_7) recently proposed by Sharma and Singh(2014). Indeed, improvement upon the difference type estimators t_3 as well as upon other estimators can be achieved when the theoretical expressions of the minimum mean squared error are considered. These expressions are based on the knowledge of population parameters which can be obtained either through past data or experience gathered in due course of time. For more discussion on this issue, the reader is referred to Das and Tripathi (1978) and Srivastava and Jhajj (1980). However, more light on this study can be focused if one would have included an empirical study. Overall this study is of academic interest as well as of practical importance, see, Diana et al. (2011), Singh et al. (2013) and Singh and Solanki (2013 a, b), Solanki and Singh(2013), Singh et al. (2013), Singh et al.(2014), Solanki et al. (2015) and Singh and Pal (2016) etc.

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