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BAYESIAN INFERENCE FOR EXPONENTIATED PARETO MODEL WITH APPLICATION TO BLADDER CANCER REMISSION TIME

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ABSTRACT

Maximum likelihood and Bayes estimators of the unknown parameters and the expected experiment times of the exponentiated Pareto model have been obtained for progressive type-II censored data with binomial removal scheme. Markov Chain Monte Carlo (MCMC) method is used to compute the Bayes estimates of the parameters of interest. The generalized entropy loss function and squared error loss function have been considered for obtaining the Bayes estimators. Comparisons are made between Bayesian and maximum likelihood (ML) estimators via Monte Carlo simulation. The proposed methodology is illustrated through real data.

Key words: PT-II CBR, MLE, bayes estimators, average experiment time.

1. Introduction

The exponentiated Pareto model (EPM) was proposed by Gupta, Gupta and Gupta (1998). The probability density function (pdf) and cumulative distribution function (cdf) of the EPM are given by

$$f(x,\alpha,\theta)=\alpha\theta\left[1-(1+x)^{-\alpha}\right]^{\theta-1}(1+x)^{-(\alpha+1)} \quad ;x>0,\alpha>0,\theta>0 \quad (1)$$

and

$$F(x,\alpha,\theta)=\left[1-(1+x)^{-\alpha}\right]^{\theta} \quad ;x>0,\alpha>0,\theta>0 \quad (2)$$

respectively, where α and θ are the shape parameters of the model. The reliability function takes the following form:

$$S(x)=1-F(x,\alpha,\theta)=1-\left[1-(1+x)^{-\alpha}\right]^{\theta},x>0,\alpha>0,\theta>0. \quad (3)$$

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A distinguished feature of EPM is that because it accommodates all types of failure rates (i.e both monotone and non-monotone). Therefore, it can be effectively used for analyzing various types of data. It may also be noted that a number of distributions can be obtained as particular cases of it. For the shape parameter $\theta = 1$, the EPM is reduced to standard Pareto distribution of second kind (see, Johnson Kotz and Balakrishnan, 1994). For more details about EPM, we refer to Gupta Gupta and Gupta (1998). Some statistical properties of this distribution and the estimators of the parameters of EPM have been discussed by Shawky and Abu-Zinadah (2009) under different estimation procedures for complete sample case. In general life testing experiments, situations do arise when units are lost or removed from the experiment while they are still functioning, i.e. we get censored data from the experiment. The loss of units may occur due to time constraints, giving type-I censored data. In such a censoring scheme, the experiment is terminated at some specified time. Sometimes, the experiment is terminated after a prefixed number of observations due to cost constraints and we get type-II censored data. The estimation of parameters of EPM has also been attempted by Afify (2010) under type-I and type-II censoring scheme. Besides the above two controlled causes, units may drop out of the experiment randomly due to some uncontrolled causes such type of situation progressive censoring arises.

For example, consider that a doctor performs an experiment with n bladder cancer patients with remission times (in months), i.e. a period during which symptoms of disease are reduced (partial remission) or disappear (complete remission) with regard to cancer, remission means there is no sign of it on scans or when the doctor examines you. Doctors use the word 'remission' instead of cure when talking about cancer because they cannot be sure that there are no cancer cells at all in the body. So the cancer could come back in the future. But the complete remission would therefore be better than partial remission. Because with partial remission the chances of occurrence of bladder cancer are higher, its means remission times (in months) are the minimum that represents partial remission, when remission times (in months) are longer, say complete remission. So the doctor performs an experiment on bladder cancer patients with partial and complete remission times (in months) are very costly and time-consuming. Due to cost constraint the experiment is terminated after a prefixed number of bladder cancer patients and we get type-II censored data. After type-II censoring another situation of bladder cancer patients with remission times (in months) may arise, the first bladder cancer patient has died due to some other unforeseen circumstances such as heart attack, accident, damage of lever, depletion of funds, etc.; some patients leave the experiment and go for treatment to other doctor/hospital. Similarly, after the second death a few more leave and so on. Finally, the doctor stops taking observation as soon as the predetermined number of deaths (say m) is recorded. Which has arise a scenario of progressive type-II censoring with random/binomial removals. For further details, readers are referred to Balakrishnan (2007). In last few years, the estimation of parameters of different life time distribution based on progressive censored samples have been studied

by several authors such as Childs and Balakrishnan (2000), Balakrishnan and Kannan (2001), Mousa and Jheen (2002), Ng, Chn and Balakrishnan (2002). The progressive type-II censoring with binomial removal were considered by Yang, Tse and Yuen (2000) for Weibull distribution, Wu and Chang (2002) for Exponential distribution. Under the progressive type-II censoring with random removals, Wu and Chang (2003) and Yuen and Tse (1996) developed the estimation problem for the Pareto distribution and Weibull distribution respectively, when the number of units removed at each failure time has a discrete uniform distribution, the expected time of this censoring plan is discussed and compared numerically. Mathematically, this experiment is similar to a life test experiment which starts with n units. At the first failure X_1, r_1 (random) units are removed randomly from the remaining $(n-1)$ surviving units. At the second failure X_2, r_2 units from remaining $n-2-r_1$ units are removed, and so on; untill m^{th} failure is observed, i.e. at m^{th} failure all the remaining $r_m = n-m-r_1-r_2\cdots r_{m-1}$ units are removed. Note that here m is pre-fixed and $r_i^{,}s$ are random. Such a censoring mechanism is termed as progressive type-II censoring with random removal scheme. If we assume that probability of removal of a unit at every stage is p for each unit then r_i can be considered to follow a binomial distribution i.e, $r_i \approx B(n-m-\sum_{l=0}^{i-1} r_l, p)$ for $i = 1, 2, 3, \cdots m-1$ and with $r_0 = 0$. The main aim of this article is concerned with the problem of obtaining Bayes estimates for the two parameter EPM based on progressive type-II censoring with binomial removals (PT-II CBR). Bayes estimators are obtained based on under square error loss function (SELF) and generalized entropy loss function (GELF). The results are obtained to PT-II CBRs, and compare the expected test times for PT-II CBR with complete sampling scheme. However, no attempt has been made to develop estimators for the parameters of EPD under PT-II CBR and its applications are discussed based on real illustration. Therefore, we propose to develop such an estimation procedure. The rest of the paper is organized as follows.

Section 2, provides the likelihood function. The ML estimators of the unknown parameters are presented in section 3. Section 4 contains the loss functions, prior distributions, the Bayes estimates using the MCMC via Gibbs sampling scheme. An algorithm for simulating the PT-II CBR is presented in section 5. We compare the expected test times under PT-II CBRs with complete sample which are given in section 6. The comparison of ML estimators and corresponding Bayes estimators are presented in section 7. These comparisons are based on simulated risk (average loss over sample space) of the estimators and discussion of results is presented. In section 8, we provide an application of the EPD distribution to remission time of bladder cancer. Finally, some conclusions are drawn in section 9.

2. Likelihood function

Let $(X_1, R_1), (X_2, R_2), (X_3, R_3), \cdots, (X_m, R_m)$ denote a progressive type-II censored sample, where $X_1 < X_2 < X_3, \cdots, X_m$.With pre-determined number of removals, say $R_1 = r_1, R_2 = r_2, R_3 = r_3, \cdots, R_m = r_m$, the conditional

likelihood function can be written as, Cohen(1963)

$$L(\alpha;\theta;x|R=r)=c^{*}\prod_{i=1}^{m}f\left(x_i\right)\left[S\left(x_i\right)\right]^{r_i}, \tag{4}$$

where $c^{*}=n(n-r_1-1)(n-r_1-r_2-2)(n-r_1-r_2-r_3-3)\cdots(n-r_1-r_2-r_3,\cdots,r_m-m+1)$, and $0\leq r_i\leq(n-m-r_1-r_2-r_3\cdots r_{i-1})$, for $i=1,2,3\ldots,m-1$. Substituting (1) and (3) into (4), we get

$$L(\alpha,\theta;x|R=r)=\prod_{i=1}^{m}\alpha\theta\left[1-(1+x_i)^{-\alpha}\right]^{\theta-1}\left\{1-\left[1-(1+x_i)^{-\alpha}\right]^{\theta}\right\}^{r_i}(1+x_i)^{-(\alpha+1)}. \tag{5}$$

Suppose that an individual unit being removed from the test at the i^{th} failure, $i=1,2,\cdots(m-1)$ is independent of the others but with the same probability p. That is the number R_i of the unit removed at i^{th} failure $i=1,2,\cdots(m-1)$ follows a binomial distribution with parameters $\left(n-m-\sum_{l=1}^{i-1}r_i,p\right)$ therefore,

$$P\left(R_1=r_1;p\right)=\binom{n-m}{r_1}p^{r_1}(1-p)^{n-m-r_1}, \tag{6}$$

and for $i=2,3,\cdots,m-1$,

$$\begin{aligned}P\left(R_{;}p\right)&=P\left(R_i=r_i|R_{i-1}=r_{i-1},\cdots R_1=r_1\right)\\&=\binom{n-m-\sum_{l=0}^{i-1}r_l}{r_i}p^{r_i}(1-p)^{n-m-\sum_{l=0}^{i-1}r_l}.\end{aligned} \tag{7}$$

Now, we further assume that R_i is independent of X_i for all i. Then, using above equations, we can write the full likelihood function as in the following form

$$L\left(\alpha,\theta,p;x,r\right)=AL_1\left(\alpha,\theta\right)L_2\left(p\right), \tag{8}$$

where

$$L_1(\alpha;\theta)=\prod_{i=1}^{m}\alpha\theta\left[1-(1+x_i)^{-\alpha}\right]^{\theta-1}\left\{1-\left[1-(1+x_i)^{-\alpha}\right]^{\theta}\right\}^{r_i}(1+x_i)^{-(\alpha+1)}, \tag{9}$$

$$L_2\left(p\right)=p^{\sum_{i=1}^{m-1}r_i}\left(1-p\right)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}. \tag{10}$$

and $A=\frac{c^{*}(n-m)!}{(n-m-\sum_{l=1}^{i-1}r_i)!\prod_{i=1}^{m-1}r_i!}$ does not depend on the parameters α,θ and p.

3. ML estimation

The ML estimations of α and θ are the simultaneous solutions of following normal equations

$$\frac{m}{\alpha}+(\theta-1)\sum_{i=1}^{m}\frac{(1+x_i)^{-\alpha}\ln(1+x_i)}{1-(1+x_i)^{-\alpha}}-\sum_{i=1}^{m}\ln(1+x_i)- \theta\sum_{i=1}^{m}\frac{r_i\left[1-(1+x_i)^{-\alpha}\right]^{\theta-1}(1+x_i)^{-\alpha}\ln(1+x_i)}{1-\left[1-(1+x_i)^{-\alpha}\right]^{\theta}}=0, \tag{11}$$

and

$$\frac{m}{\theta}+\sum_{i=1}^{m}\ln\left[1-(1+x_i)^{-\alpha}\right] \sum_{i=1}^{m}\frac{r_i\left[1-(1+x_i)^{-\alpha}\right]^{\theta}\ln\left[1-(1+x_i)^{-\alpha}\right]}{1-\left[1-(1+x_i)^{-\alpha}\right]^{\theta}}=0. \tag{12}$$

It may be noted that (11) and (12) cannot be solved simultaneously to provide a nicely closed form for the estimators. Therefore, we propose to use fixed point iteration method for solving these equations. For details about the proposed method readers may refer to Jain, Iyengar and Jain (1984).

4. Bayesian estimation

This section is concerned with prior distributions for unknown parameters, symmetric and asymmetric loss function and Bayes estimates using the Gibbs sampling scheme.

Prior and posterior distributions

In order to obtain the Bayes estimators of unknown parameters α and θ based on PT-II CBRs, we must assume that the parameters α and θ are random variables. The model under consideration has shapes and censoring parameters, and continuous conjugate priors for these parameters do not exist. We further assume that these random variables α and θ have independently distributed informative prior distribution with respective prior pdfs

$$g_1(\alpha)=\frac{{\lambda_1}^{\nu_1}e^{-\lambda_1\alpha}\alpha^{\nu_1-1}}{\Gamma\nu_1} \quad ; \quad 0<\alpha<\infty, \quad \lambda_1>0, \quad \nu_1>0 \tag{13}$$

$$g_2(\theta)=\frac{{\lambda_2}^{\nu_2}e^{-\lambda_2\theta}\theta^{\nu_2-1}}{\Gamma\nu_2} \quad ; \quad 0<\theta<\infty, \quad \lambda_2>0, \quad \nu_2>0 \tag{14}$$

respectively. But Arnold and Press (1983) had all ready discussed that there is no clear cut way in which one can say that one prior is better than the other. But for purpose of Bayesian analysis, the gamma prior $g_1(\alpha)$ and $g_2(\theta)$ are chosen instead of the exponential prior of α and θ used by Eissa and Nassar (2004) and Jung, Chung and Kim (2011) because the gamma prior is wealthy enough to cover the prior belief of the experimenter for different shapes. On the basis of the above stated assumptions, the joint prior pdf of α and θ is

$$g(\alpha,\theta) = g_1(\alpha)\, g_2(\theta) \quad ; \quad \alpha > 0, \quad \theta > 0 \tag{15}$$

Combining the priors given by (13) and (14) with likelihood given by (9), we can easily obtain joint posterior pdf of (α,θ) as $\pi(\alpha,\theta|x,r) = \frac{J_1}{J_0}$ where

$$\begin{aligned} J_1 = \alpha^{m+\nu_1-1}\theta^{m+\nu_2-1} e^{-\left(\sum_{i=1}^m \lambda_1\alpha + \sum_{i=1}^m \lambda_2\theta\right)} \prod_{i=1}^m \left[1-(1+x_i)^{-\alpha}\right]^{\theta-1} \\ \left\{1-\left[1-(1+x_i)^{-\alpha}\right]^{\theta}\right\}^{r_i} (1+x_i)^{-(\alpha+1)}, \end{aligned} \tag{16}$$

and $J_0 = \int_0^\infty \int_0^\infty J_1 d\alpha d\theta$. Hence, the respective marginal posterior pdfs of α and θ are given by

$$\pi_1(\alpha|x,r) = \int_0^\infty \frac{J_1}{J_0} d\theta, \tag{17}$$

and

$$\pi_2(\theta|x,r) = \int_0^\infty \frac{J_1}{J_0} d\alpha. \tag{18}$$

Loss functions

In order to select the best decision in decision theory, an appropriate loss function must be specified. For this purpose, we use symmetric as well as asymmetric loss function. The Bayes estimators are obtained under SELF

$$l_1(\phi,\hat{\phi}) = \in_1 \left(\phi - \hat{\phi}\right)^2 ; \quad \in_1 > 0 \tag{19}$$

where $\hat{\phi}$ is the estimate of the parameter ϕ and the Bayes estimator $\hat{\phi}_S$ of ϕ comes out to be $E_\phi[\phi]$, where E_ϕ denotes the posterior expectation. However, this loss function is symmetric loss function and can only be justified if over-estimation and under-estimation of equal magnitudes are of equal seriousness. A number of asymmetric loss functions are also available in the statistical literature. Let us consider the GELF, proposed by Calabria and Pulcini (1996), defined as follows :

$$l_2(\phi,\hat{\phi}) = \in_2 \left(\left(\frac{\hat{\phi}}{\phi}\right)^{\delta} - \delta \ln\left(\frac{\hat{\phi}}{\phi}\right) - 1 \right) ; \quad \in_2 > 0 \tag{20}$$

The constant δ, involved in (20), is its shape parameter. It reflects departure from symmetry. When $\delta > 0$, it considers over-estimation (i.e., positive error) to be more serious than under-estimation (i.e., negative error) and converse for $\delta < 0$. The Bayes estimator $\hat{\phi}_E$ of ϕ under GELF is given by

$$\hat{\phi}_E = \left[E_\phi\left(\phi^{-\delta}\right)\right]^{\left(-\frac{1}{\delta}\right)} \tag{21}$$

provided the posterior expectation exits. It may be noted here that for $\delta = -1$, the Bayes estimator under loss (19) coincides with the Bayes estimator under SELF l_1. Expressions for the Bayes estimators $\hat{\alpha}_E$ and $\hat{\theta}_E$ for α and θ respectively, under GELF can be given as

$$\hat{\alpha_E} = \left[\int_0^\infty \alpha^{-\delta}\pi_1\left(\alpha|x,r\right)d\alpha\right]^{\left(-\frac{1}{\delta}\right)}, \tag{22}$$

and

$$\hat{\theta_E} = \left[\int_0^\infty \theta^{-\delta}\pi_1\left(\theta|x,r\right)d\theta\right]^{\left(-\frac{1}{\delta}\right)}, \tag{23}$$

It is to mention here that from equation (22) and (23), the Bayes estimators $\hat{\alpha}_E$ and $\hat{\theta}_E$ are not reducible in a nice closed form. Therefore, we use the numerical techniques for obtaining the estimates. We, therefore, propose to consider Gibbs sampling procedure.

MCMC method via Gibbs sampling

In this subsection, we use the Gibbs sampling procedure to obtain the Bayes estimates α and θ under SELF and GELF. It is clear from equations (22) and (23) that the Bayes estimators of α and θ are not obtained analytically and numerical techniques must be used for computations. To compute Bayes estimators of the parameters α and θ we propose to use MCMC technique, via Gibbs sampler along with Metropolis-Hastings algorithms to generate samples from posterior distributions and then compute Bayes estimates. The Gibbs is an algorithm for simulating from the full conditional posterior distributions while the Metropolis-Hastings algorithm generates samples from an (essentially) arbitrary proposal distribution. For more details about the MCMC methods see, for example, Vasishta, Smith and Upadhyay (2001) and Gupta and Upadhyay (2010). The full conditional posterior distributions of the parameters α and θ are, respectively, given as

$$\tau_1(\alpha|x,r) \propto \alpha^{m+\nu_1-1}e^{-\left(\sum_{i=1}^m \lambda_1\alpha\right)}\prod_{i=1}^m\left[1-(1+x_i)^{-\alpha}\right]^{\theta-1}\left\{1-\left[1-(1+x_i)^{-\alpha}\right]^{\theta}\right\}^{r_i} \tag{24}$$
$$(1+x_i)^{-(\alpha+1)}$$

$$\tau_2(\theta|x,r) \propto \theta^{m+\nu_2-1}e^{-\left(\sum_{i=1}^m \lambda_2\theta\right)}\prod_{i=1}^m\left[1-(1+x_i)^{-\alpha}\right]^{\theta-1}\left\{1-\left[1-(1+x_i)^{-\alpha}\right]^{\theta}\right\}^{r_i} \tag{25}$$

The following MCMC algorithm is used to generate the posterior samples and then to obtain the Bayes estimates of α and θ.

I. Start with initial guesses of α and θ say α_0 and θ_0.
II. Set j=1.
III. Generate α_1 from $\tau_1(\alpha|\theta, \mathbf{x}, r)$ and θ_1 from $\tau_2(\theta|\alpha, \mathbf{x}, r)$.
IV. Repeat steps 2-3, N times.
V. Now, the Bayes estimates of α and θ under GELF are, respectively, given as

$$\hat{\alpha}_E = \left[\frac{1}{N-M} \sum_{j=M+1}^{N} {\alpha_j}^{-\delta} \right]^{-1/\delta} \tag{26}$$

$$\hat{\theta}_E = \left[\frac{1}{N-M} \sum_{j=M+1}^{N} {\theta_j}^{-\delta} \right]^{-1/\delta} \tag{27}$$

VI. Put $\delta = -1$ in above step 5, then the Bayes estimator under GELF coincides with Bayes estimator under SELF.
where M is the burn-in period (i.e, the number of iterations before the stationary distribution is achieved).

5. Algorithm for PT-II CBR

We need to simulate PT-II CBR from specified EPD. To get such a sample, we propose the use of the following algorithm:

I. Specify the value of n.
II. Specify the value of m.
III. Specify the value of parameters α, θ and p.
IV. Generate random number r_i from B$\left(n - m - \sum_{l=0}^{i-1} r_l, p\right)$, for $i = 1, 2, 3, \cdots, m-1$.
V. Set r_m according to the following relation.
VI. $r_m = \begin{cases} n - m - \sum_{l=1}^{m-1} r_l & \text{if } n - m - \sum_{l=1}^{m-1} r_l > 0 \\ 0 & \text{otherwise} \end{cases}$
VII. Generate m independent $U(0,1)$ random variables $W_1, W_2, \cdots, W_m$.
VIII. For given values of the progressive type-II censoring scheme $r_i (i = 1, 2, \cdots, m)$ set $V_i = W_i^{1/(i+r_m+\cdot+r_{m-i+1})} (i = 1, 2, \cdots, m)$.
IX. Set $U_i = 1 - V_m V_{m-1} \cdots V_{m-i+1} (i = 1, 2, \cdots, m)$, then $U_1, U_2, \cdots, U_m$ are progressive type-II censored samples with binomial removals of size m from $U(0,1)$.
X. Finally, for given values of parameters α and λ, we set $x_i = F^{-1}(U) (i = 1, 2, \cdots, m)$. Then,$(x_1, x_2, \cdots, x_m)$ is the required from progressive censoring with binomial removals sample of size m from the EPD.

6. Average Experiment Time

In practical situations, an Experimenter may be interested to know whether the test can be completed within a specified time. This information is important for an experimenter to choose an appropriate sampling plan because the time required to complete a test is directly related to cost. Under Progressive censoring with a fixed number of removal the time is given by X_m. According to Balakrishnan and Aggarwalla (2000), the expected value of X_m is given by

$$E\left[X_m|R\right] = C(r)\sum_{l_1=0}^{r_1}\sum_{l_2=0}^{r_2}\cdots\sum_{l_m=0}^{r_m}(-1)^B\frac{C_{l_1=0}^{r_1}\cdots C_{l_m=0}^{r_m}}{\prod_{i=1}^{m-1}h(l_i)}\int_0^\infty xf(x)F^{h(l_m)-1}(x)\partial x. \tag{28}$$

where $B=\sum_{i=1}^{m}l_i$, $h(l_i)=l_1+l_2+\cdots+l_i+i$, $C(r)=n(n-r_1-1)(n-r_1-r_2-2)\cdots[n-\sum_{i=1}^{m-1}(r_i+1)]$ and i is the number of live units removed from experiment (number of failure units). Using the p.d.f and c.d.f of EPD, the equation will be

$$E\left[X_m|R\right] = C(r)\sum_{l_1=0}^{r_1}\sum_{l_2=0}^{r_2}\cdots\sum_{l_m=0}^{r_m}(-1)^B\frac{C_{l_1=0}^{r_1}\cdots C_{l_m=0}^{r_m}}{\prod_{i=1}^{m-1}h(l_i)}$$
$$\int_0^\infty x_i\alpha\theta\left[1-(1+x)^{-\alpha}\right]^{\theta-1}(1+x)^{-(\alpha+1)}\left\{\left[1-(1+x)^{-\alpha}\right]^{\theta}\right\}^{(h(l_m)-1)} \tag{29}$$

Let

$$\begin{aligned}
S_1 &= \alpha\theta\int_0^\infty x_i\left[1-(1+x)^{-\alpha}\right]^{\theta-1}(1+x)^{-(\alpha+1)}\left\{\left[1-(1+x)^{-\alpha}\right]^{\theta}\right\}^{(h(l_m)-1)}\\
&= \alpha\theta\int_0^\infty x_i(1+x)^{-(\alpha+1)}\left[1-(1+x)^{-\alpha}\right]^{(h(l_m)\theta-1)}\partial x_i.\\
&= \alpha\theta\sum_{k=0}^{h(l_m)\theta-1}(-1)^k\binom{h(l_m)\theta-1}{k}\int_0^\infty\frac{x_i}{(1+x_i)^{(\alpha(k+1)+1)}}\partial x_i\\
&= \alpha\theta\sum_{k=0}^{h(l_m)\theta-1}(-1)^k\binom{h(l_m)\theta-1}{k}B_{II}(2,\alpha(k+1)-1)
\end{aligned}$$

Putting this value in to the right hand of equation (29), the expected test time is given by

$$E\left[X_m|R\right] = C(r)\alpha\theta \sum_{l_1=0}^{r_1}\sum_{l_2=0}^{r_2}\cdots\sum_{l_m=0}^{r_m}(-1)^B \frac{C_{l_1=0}^{r_1}\cdots C_{l_m=0}^{r_m}}{\prod_{i=1}^{m-1}h(l_i)} \\ \alpha\theta \sum_{k=0}^{h(l_m)\theta-1}(-1)^k \binom{h(l_m)\theta-1}{k} B_{II}(2,\alpha(k+1)-1) \tag{30}$$

The expected test time for PT-II CBRs is evaluated by taking expectation on both sides (29) with respect to the R. That is

$$\begin{aligned} E\left[X_m\right] &= E_R\left[E\left[X_m|R=r\right]\right] \\ &= \sum_{r_1=0}^{g(r_1)}\sum_{r_2=0}^{g(r_2)}\cdots\sum_{r_{m-1}=0}^{g(r_{m-1})} P(R,p)E\left[X_m|R=r\right]. \end{aligned} \tag{31}$$

where $g(r_i) = n-m-r_1-\cdots-r_{i-1}$ and $P(R;p)$ is given in equation (7). For the expected time a complete sampling case with n test units is obtained by taking $m=n$ and $r_i=0$ for all $i=1,2,\cdots,m$, in (30). We have

$$E\left[X_n^*\right] = n\alpha\theta\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k B_{II}(2,\alpha(k+1)-1). \tag{32}$$

Also, the expected time of a type-II censoring without removal is defined by the expected value of the m^{th} failure time, then

$$E\left[X_m^*\right] = m\alpha\theta\binom{n}{m}\sum_{k=0}^{m-1}\binom{m-1}{k}(-1)^k B_{II}(2,\alpha(k+1)-1), \tag{33}$$

The ratio of the expected experiment time (REET)δ_{REET} is computed between PT-II CBR and the complete sampling, we define

$$\delta_{REET} = \frac{E[X_m]\ Under\ PT-II\ CBR}{E[X_n{}^*]\quad under\quad complete\quad sampling}. \tag{34}$$

It can be noted from δ_{REET} that important information is given in order to determine significantly the shortest experiment time if a much larger sample of n test units is used, the test is terminated, when m^{th} failures have been observed. But here we are interested in considering various values of n, m and p, numerically calculated under the expected experiment time of PT-II CBR and complete sample, which are derived in equations (31) and (32). Numerical results are obtained in Table 7 where for $n = 15, 12$ and 9 corresponding choices of m are given. From Table 7 we observed that, when n is fixed, the values of the δ_{REET} and expected termination time under PT-II CBR and complete test decrease as m decreases, while for fixed m, the value of the δ_{REET} and expected termination time under PT-II CBR and complete sampling increase as n decreases. Finally, for fixed values of m and n, we

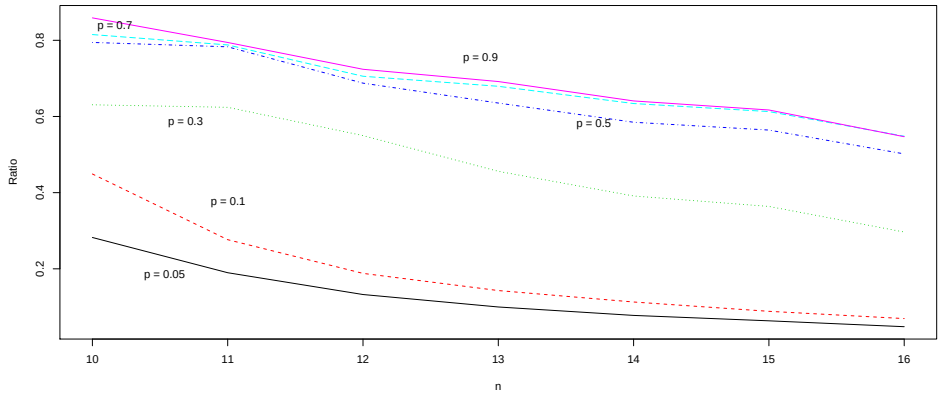

FIGURE 1. δ_{REET} under PT-II CBRs to δ_{REET} under complete sample

observed that the effect of variation of removal probability p with the values of the δ_{REET} and expected termination time of PT-II CBR increase as p increases.

Figure 1 shows the ratio of the expected test time under PT-II CBR to the expected test time under complete sample versus n for $m = 8$ and different values of removal probability p. We observed that, when the value of p is large, the ratio increases and approaches 1 quickly and the expected test time is not small in these cases. Hence, for small p, the expected test time is more significant than larger value of p. So, we have taken $p = 0.3$ from Figure 1, which was significant for further calculation.

7. Simulation studies

The estimators $\hat{\alpha}_M$ and $\hat{\theta}_M$ denote the ML estimators of the parameters α and θ respectively while $\hat{\alpha}_S$ and $\hat{\theta}_S$ are corresponding Bayes estimators under SELF and $\hat{\alpha}_E$ and $\hat{\theta}_E$ are the corresponding Bayes estimators under GELF. We compare the estimators obtained under GELF with corresponding ML estimators and Bayes estimators under SELF. The comparisons are based on the simulated risks (average loss over sample space) under GELF. It may be mentioned here that the exact expressions for the risks cannot be obtained because estimators are not in a nice closed form. Therefore, the risks of the estimators are estimated on the basis of Monte-carlo simulation study of 10000 samples. It may be noted that the risks of the estimators will depend on values of $n, m, \theta, \alpha, p, \lambda_1, \lambda_2, \nu_1, \nu_2$ and δ. In order to consider variation in the values of these parameters, we have obtained the simulated risks for $m = 9\,[3]\,15$, when $n = 15$, $\theta = 0.5$, $\alpha = 2$, $\delta = \pm 0.5$ and $p = 0.3$. For prior distribution we have used non-informative prior with $\lambda_1 = \lambda_2 = \nu_1 = \nu_2 = 0$, and informative prior and the hyper parameter are chosen in such a way that the prior

mean became true value of the parameter and belief in prior mean strong or weak, i.e. the prior variance is small and large. Thus, the values of the hyper parameter of informative prior are $\lambda_1 = (0.5, 4), \lambda_2 = (0.125, 1), \nu_1 = (1, 8), \nu_2 = (0.0625, 0.5)$. Generating the progressive sample as mentioned in section 4, the simulated risks under SELF and GELF have been obtained for different values of m with selected values of the rest of the parameters $n, \theta, \alpha, p, \lambda_1, \lambda_2, \nu_1, \nu_2$ and δ have been taken. The results are given in tables Table 1-6. The entries in brackets in all the tables denote the risks of the estimators when δ is negative and the other non-bracket entries are the risks when δ is positive.

Discussion of the results

It is interesting to note that when effective sample size m increases, keeping n, fixed for fixed positive value of δ under both losses, the risks of the ML estimate of α, first increase then decrease slightly as m increases whereas the risks of Bayes estimators always increase with the increase in the value of m. This trend of the magnitude of the risks is also the same for fixed negative value of δ. It is observed when non-informative prior for α has been used (see, Table 1). While regarding the considered prior distribution, when we have smaller belief in considered prior distribution for α, i.e. prior variance is 1, then we observe that in over-estimation situation under both losses, the risks of estimator $\hat{\alpha}_M$ increase then slightly decrease as m increases but in under estimation situation under both losses, the risks of estimator of $\hat{\alpha}_M$ decrease then slightly increase as m increases. Finally, we observed that under both losses for positive and negative values of δ, the risks of estimator of $\hat{\alpha}_S$ and $\hat{\alpha}_E$ increase as m increases (see, Table 2). For larger prior variance of α, we observed that under both losses for $\delta < 0$, the risks of estimator $\hat{\alpha}_M$ decrease as m increases, and the rest of them for $\delta < 0$ and $\delta > 0$, the risks of estimators $\hat{\alpha}_S$, $\hat{\alpha}_E$ and for $\delta > 0$ $\hat{\alpha}_M$ increase as m increases (see, Table 3). The risk of estimators of θ under SELF and GELF, when priors for the parameter θ are non-informative types, the risks of estimator $\hat{\theta}_M$, decrease in case of both positive and negative values of δ, and the risks of Bayes estimators increase as m increases for both positive and negative values of δ, and under both losses namely SELF and GELF (see, Table 4). For smaller prior variance of θ, we observed under both losses that when $\delta > 0$, the risk of estimator $\hat{\theta}_M$ decreases as m increases but when $\delta < 0$, the risk of estimator $\hat{\theta}_M$ first increases then decreases as m increases and as in the previous table the risk of Bayes estimators as m increases for both positive and negative values of δ under both losses. The risk of estimators of θ under SELF and GELF, when prior for the parameter θ are non informative types, the risks of estimator $\hat{\theta}_M$ decrease in case of both positive and negative values of δ and the risks of Bayes estimators increase as m increases for both positive and negative values of δ and under both losses, namely SELF and GELF (see, Table 5). For larger prior variance of θ, under both losses, for $\delta > 0$ and $\delta < 0$, the risk of estimator $\hat{\theta}_M$ decreases as m increases and as in the previous cases, the trends for the risks are the same (see, Table 6).

8. Application

In this section we reanalyze the data extracted from Luz, Silva, Rodrigo, Bouruignon, Andrea and Gauss Coreiro (2012). For the purpose of real illustration, we have been discussed in presence of PT-II CBR. The data describe a study of remission time(in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003). The data are given as

0.08,2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26,3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54,3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77,32.15, 2.64, 3.88,5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34,7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23,5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26,2.83, 4.33, 5.49,7.66,11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79,18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

In order to identify the shape of lifetime data failure rate function, we shall consider, as a crude indicative, a graphical method based on TTT (Total time on test) plot Aarset (1985) Hence, in its empirical version the TTT plot is given as

$$T(\tfrac{n}{r}) = \frac{\sum_{i=1}^{r} y_{(i)} + (n-r)y_{(r)}}{\sum_{i=1}^{n} y_{(i)}}$$

where $r = 1, 2, \cdots, n$ and $y_{(r)}$ is the order statistics of the sample. On the basis of TTT plot, we identify that the failure rate function is increasing, decreasing and increasing then decreasing, i.e. when the TTT plot for considered data is concave, convex and concave then convex respectively. Figure 2 shows that TTT plot for considered data, which is concave then convex indicating an increasing then decreasing failure rate function, is properly accommodated by EPD with increasing then decreasing failure rate. According to Figure 3, we observed that this data is appropriate for EPD and Figure 4 shows estimated pdf, CDF and hazard functions. Also, we have obtained Kolmogrov-Smirnov (K-S) Statistics, Akaike's information criterion (AIC) and Bayesian information criterion (BIC) under sub model Pareto distribution for given data set and values summarized in Table 8. According to above considered criterion, we can say that EPD provide better fit than Pareto distribution. Therefore, we use this data to illustrate the proposed methodology. For this PT-II CBRs are generated from the given data under various schemes, which are summarized in Table 11. We have obtained the ML estimates, Bayes estimates (using non-informative prior), 95% CI and HPD intervals for the parameters α and θ respectively under SELF and GELF for $\delta = \pm 1.5$, and value of the hyper parameters α and θ are taken as $\nu_1 = 0.00001, \lambda_1 = 0.0001$ and $\nu_2 = 0.00001, \lambda_2 = 0.0001$ respectively. We have obtained the ML and the Bayes estimates of α and θ under SELF and GELF for $\delta = \pm 1.5$ presented in Table 9 and 10 respectively. When the degree of censoring decreases, the estimate of α and θ is closer to the estimates of without censoring. Under different censoring schemes, the length of HPD intervals is always less than

CI. The ML and Bayes estimates under SELF and GELF of α and θ always lies between HPD and CI.

9. Conclusion

In this paper, we consider a Bayesian estimation of EPD in presence of PT-II CBRs under the asymmetric loss function. We use independent gamma priors for the unknown parameters as the continuous conjugate priors do not exist. It is seen that the explicit expressions for the Bayes estimators are not possible. We obtain the approximate Bayes estimates of parameters using the MCMC via Gibbs sampling scheme. To observe the properties of the Bayes estimators based on the MCMC via Gibbs sampling, some numerical experiments are performed. In general most of cases, when the sample size increases the risk of the estimators decreases. The interesting points are observed regarding PT-II CBR, either prior belief of the model parameter is low or high, our proposed estimators $\hat{\alpha}_E$ and $\hat{\theta}_E$ perform well (in the sense of having smaller risk).

On the other hand, in context of the expected experiment time, we may also conclude that the removal probability p plays a great role in the expected test time. The increase in the removal probability p means more items are removed at the early stage of the experiment. Hence, for larger p, the collection of observations much closer to the tail of the life time distribution and the experiment under PT-II CBR increase as p increases.

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REFERENCES

AARSET, M. W., (1985). The null distribution for a test of constant versus bathtub failure rate. *Scandinavian Journal of Statistics*, 12(1):55-68.

AFIFY, W. M., (2010). On estimation of the exponentiated Pareto distribution under different sample scheme. *Applied Mathematical Sciences*, 4(8):393–402.

ARNOLD, B. C., PRESS, S. J., (1983). Bayesian inference for Pareto populations. *J.Econom.*, 21:287-306.

BALAKRISHNAN, N., (2007). Progressive methodology: An appraisal (with discussion). *Test*, 16 (2):211–259.

BALAKRISHNAN, N., AGGARWALLA, R., (2000). *Progressive Censoring: Theory, Methods and Applications*. Birkhauser, Boston.

BALAKRISHNAN, N., KANNAN, N., (2001). Point and Interval Estimation for Parameters of the Logistic Distribution Based on Progressively Type-II Censored Samples, in Handbook of Statisticsm N. Balakrishnan and C. R. Rao, 20. Eds. Amsterdam, North-Holand.

CALABRIA, R., PULCINI, G., (1996). Point estimation under-asymmetric loss functions for life-truncated exponential samples. *Commun. statist. Theory meth.*, 25(3):585–600.

CHILDS, A., BALAKRISHNAN, N., (2000). Conditional inference procedures for the Laplace distribution when the observed samples are Progressively censored. *Metrika*, 52:253–265.

COHEN, A. C., (1963). Progressively censored samples in life testing. *Technometrics*, pages 327–339.

EISSA, F. H., NASSAR, M. M.,(2004). Bayesian estimation for the exponentiated Weibull model. *Communication in Statistics Theory and Methods*, 33:2343–2236.

GUPTA, R. C., GUPTA, R. D., GUPTA, P. L., (1998). Modeling failure time data by Lehman alternatives. *Commun. Statist. - Theory Meth.*, 27(4):887–904.

GUPTA, A., UPADHYAY, S. K., (2010). A Bayes analysis of modified Weibull distribution via Markov chain monte carlo simulation. *Journal of Statistical Computation and Simulation*, 80(3):241–254.

JAIN, M. K., IYENGAR, S. R. K., JAIN, R. K.,(1984). *Numerical Methods for Scientific and Engineering Computation*. New Age International (P) Limited, Publishers, New Delhi, fifth edition.

JOHANSON, N. L., KOTZ, S., BALAKRISHNAN, N., (1994). *Continuous Univariate Distributions*, volume 1. Wiley, New York, 2 edition.

JUNG, J., CHUNG, Y., KIM, C.,(2011). Bayesian estimation for the exponentiated Weibull model under type II progressive censoring. *Statistical Papers (accepted)*.

LEE, E. T., WANG, J. W.,(2003). *Statistical Methods for Survival Data Analysis*. Wiley, New York, 3rd edition.

LUZ, M. ZEA, SILVA RODRIGO, B., BOURGUIGNON, M., ANDREA, S., GAUSS COREIRO, M., (2012). The Beta Exponentiated Pareto Distribution with Application to Bladder Cancer Susceptibility. *International Journal of Statistics and Probability*, 1(2):8–19.

MOUSA, M., JAHEEN, Z., (2002). Statistical inference for the burr model based on progressively censored data. *An International Computers and Mathematics with Applications,*, 43:1441–1449.

NG, K., CHAN, P. S.,BALAKRISHAN, N.,(2002). Estimation of parameters from progressively censored data using an algorithm. *Computational Statistics and Data Analysis*, 39:371–386.

SHAWKY, A. I., HANNA, H. ABU-ZINADAH.,(2009). Exponentiated Pareto distribution: Different method of estimations. *Int. J.Contemp. Math. Sciences*, 4(14): 677–693.

VASISHTA, N., SMITH, A. F. M., UPADHYAY, S. K., (2001). Bayes inference in life testing and reliability via Markov chain Monte Carlo simulation. *Sankhya*, A 63(1):15–20.

WU, S. J., CHANG, C. T. (2002). Parameter estimations based on exponential progressive type II censored with binomial removals. *International Journal of Information and Management Sciences*, 13:37–46.

WU, S. J., CHANG, C. T., (2003). Inference in the Pareto distribution based on progressive type II censoring with random removales. *Journal of Applied Statistics*, 30:163–172.

YANG, C., TSE, S. K., YUEN, H. K., (2000). Statistical analysis of Weibull distributed life time data under type II progressive censoring with binomial removals. *Jounal of Applied Statistics*, 27:1033–1043.

YUEN, H. K., TSE, S. K., (1996). Parameters estimation for Weibull distribution under progressive censoring with random removal. *Journal Statis. Comput. Simul*, 55:57–71.

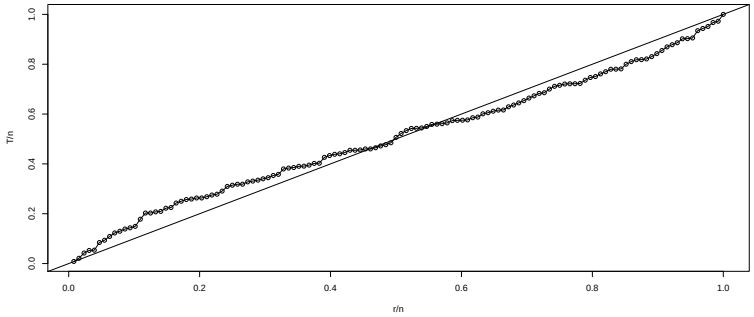

FIGURE 2. TTT plot for the remission times (in months) of 128 bladder cancer patients

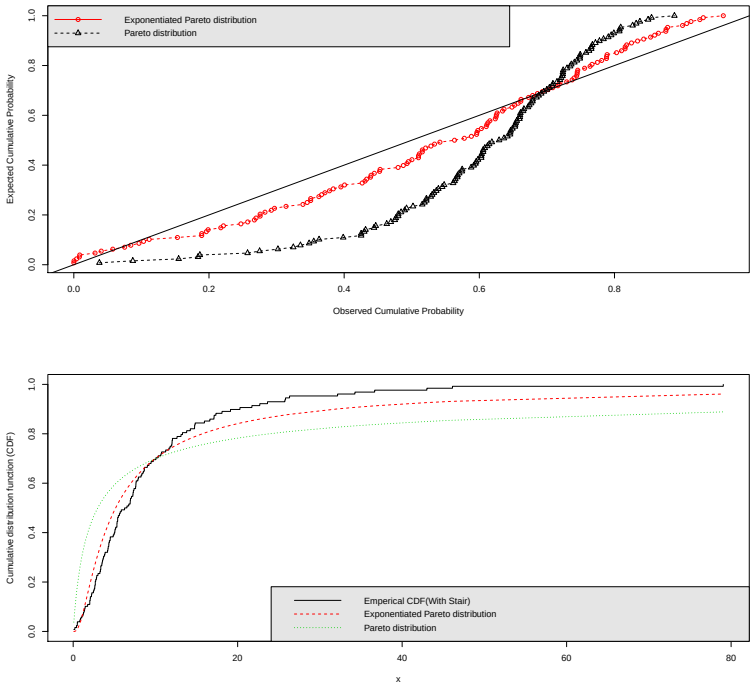

FIGURE 3. Upper graph represents the probability plot and lower graph shows CDF plot for the remission times (in months) of 128 bladder cancer patients.

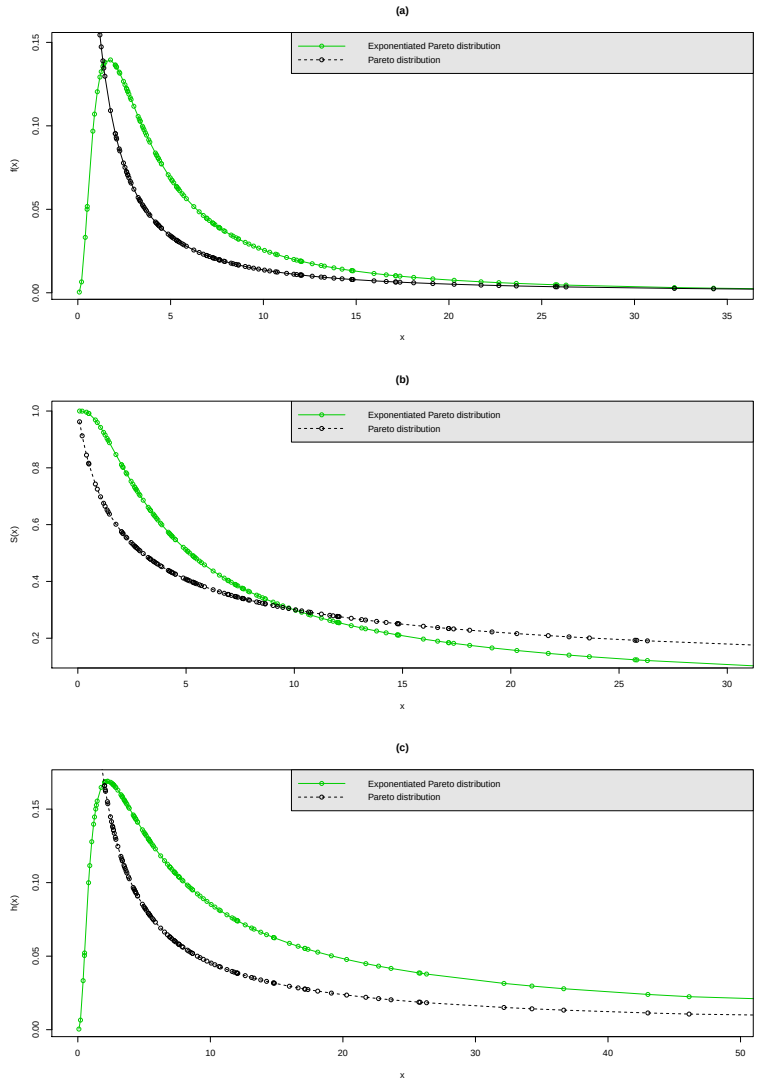

FIGURE 4. Estimated probability density, survival and hazard functions for the remission times (in months) of 128 bladder cancer patients.

TABLE 1. Risks of estimators of α under different losses for fixed $\alpha = 2$, $\theta = 0.5$, $\nu_1 = 0$, $\lambda_1 = 0$, $\nu_2 = 0$, $\lambda_2 = 0$, n= 15, $\delta = \pm 0.5$

m	$R_S(\hat{\alpha}_M)$	$R_S(\hat{\alpha}_S)$	$R_S(\hat{\alpha}_E)$	$R_E(\hat{\alpha}_M)$	$R_E(\hat{\alpha}_S)$	$R_E(\hat{\alpha}_E)$
9	0.70526	0.41143	.021448	0.01624	0.010061	0.000743
	(0.71638)	(0.42280)	(0.19942)	(0.01462)	(0.00937)	(0.00484)
12	0.72537	0.50554	0.06051	0.01665	0.01211	0.001691
	(0.72006)	(0.50812)	(0.33002)	(0.01468)	(0.01096)	(0.00757)
15	0.72402	0.54560	0.13901	0.01662	0.01296	0.00372
	(0.71659)	(0.53729)	(0.39171)	(0.01462)	(0.01150)	(0.00879)

TABLE 2. Risks of estimators of α under different losses for fixed $\alpha = 2$, $\theta = 0.5$, $\nu_1 = 4$, $\lambda_1 = 2$, n=15, $\nu_2 = 0.25$, $\lambda_2 = .5$, $\delta = \pm 0.5$

m	$R_S(\hat{\alpha}_M)$	$R_S(\hat{\alpha}_S)$	$R_S(\hat{\alpha}_E)$	$R_E(\hat{\alpha}_M)$	$R_E(\hat{\alpha}_S)$	$R_E(\hat{\alpha}_E)$
9	0.71061	0.07906	0.00245	0.01635	0.00220	.00074
	(0.70607)	(0.07960)	(0.04193)	(0.01445)	(0.00212)	(0.00116)
12	0.72158	0.13124	0.02286	0.01657	0.00355	0.00067
	(0.70378)	(0.12874)	(0.08569)	(0.01441)	(0.00330)	(0.00227)
15	0.70846	0.16820	0.04939	0.01631	0.00447	0.001406
	(0.71687)	(0.16936)	(0.12436)	(0.01463)	(0.00422)	(0.00319)

TABLE 3. Risks of estimators of α under different losses for fixed $\alpha = 2$, $\theta = 0.5$, $\nu_1 = 1$, $\lambda_1 = 0.5$, n=15, $\nu_2 = 0.0625$, $\lambda_2 = 0.125$, n=15, $\delta = \pm 0.5$

m	$R_S(\hat{\alpha}_M)$	$R_S(\hat{\alpha}_S)$	$R_S(\hat{\alpha}_E)$	$R_E(\hat{\alpha}_M)$	$R_E(\hat{\alpha}_S)$	$R_E(\hat{\alpha}_E)$
9	0.69846	0.23747	0.00534	0.01610	0.00613	0.00016
	(0.71844)	(0.24831)	(0.12497)	(0.01465)	(0.00592)	(0.00319)
12	0.71230	0.32581	0.04418	0.01638	0.00818	0.00125
	(0.71827)	(0.32775)	(0.21453)	(0.01465)	(0.00755)	(0.00520)
15	0.72212	0.38485	0.10290	0.01658	0.00951	0.00281
	(0.71713)	(0.38264)	(0.27947)	(0.01464)	(0.00864)	(0.00657)

TABLE 4. Risks of estimators of θ under different losses for fixed $\alpha = 2$, $\theta = 0.5$, $\nu_1 = 0$, $\lambda_1 = 0$, $\nu_2 = 0$, $\lambda_2 = 0$, n= 15, $\delta = \pm 0.5$

m	$R_S(\hat{\theta}_M)$	$R_S(\hat{\theta}_S)$	$R_S(\hat{\theta}_E)$	$R_E(\hat{\theta}_M)$	$R_E(\hat{\theta}_S)$	$R_E(\hat{\theta}_E)$
9	0.02434	0.01574	0.00451	0.00957	0.00642	0.00199
	(0.02350)	(0.01526)	(0.01082)	(0.008449)	(0.00577)	(0.00424)
12	0.02336	0.01687	0.00621	0.00921	0.00685	0.00269
	(0.02336)	(0.01717)	(0.01299)	(0.00834)	(0.00641)	(0.00500)
15	0.02256	0.01712	0.00699	0.00893	0.00693	0.00301
	(0.02305)	(0.01739)	(0.01351)	(0.00831)	(0.00649)	(0.00518)

TABLE 5. Risks of estimators of θ under different losses for fixed $\alpha = 2$, $\theta = 0.5$, $\nu_1 = 4$, $\lambda_1 = 2$, n=15, $\nu_2 = 0.25$, $\lambda_2 = .5$, $\delta = \pm 0.5$

m	$R_S(\hat{\theta}_M)$	$R_S(\hat{\theta}_S)$	$R_S(\hat{\theta}_E)$	$R_E(\hat{\theta}_M)$	$R_E(\hat{\theta}_S)$	$R_E(\hat{\theta}_E)$
9	0.02402	0.009106	0.00303	0.00945	0.00387	0.00136
	(0.02359)	(0.00911)	(0.00670)	(0.008478)	(0.00364)	(0.00275)
12	0.0230799	0.00939	0.00339	0.00912	0.00399	0.00151
	(0.02405)	(0.01004)	(0.00761)	(0.00861)	(0.00397)	(0.00309)
15	0.02300	0.00964	0.00375	0.00909	0.00409	0.00167
	(0.0224)	(0.00932)	(0.00712)	(0.00811)	(0.00371)	(0.00290)

TABLE 6. Risks of estimators of θ under different losses for fixed $\alpha = 2$, $\theta = 0.5$, $\nu_1 = 1$, $\lambda_1 = 0.5$, n=15, $\nu_2 = 0.0625$, $\lambda_2 = 0.125$, $\delta = \pm 0.5$

m	$R_S(\hat{\theta}_M)$	$R_S(\hat{\theta}_S)$	$R_S(\hat{\theta}_E)$	$R_E(\hat{\theta}_M)$	$R_E(\hat{\theta}_S)$	$R_E(\hat{\theta}_E)$
9	0.02446	0.01318	0.004151	0.00961	0.005469	0.00184
	(0.02434)	(0.0131)	(0.00953)	(0.008710)	(0.00505)	(0.0037)
12	0.02321	0.01385	0.00510	0.00916	0.00571	0.00223
	(0.02338)	(0.01392)	(0.0105)	(0.00841)	(0.00533)	(0.00414)
15	0.02298	0.01438	0.00582	0.009078	0.00591	0.00253
	(0.02272)	(0.01419)	(0.01095)	(0.008199)	(0.005418)	(0.00429)

TABLE 7. Expected experiment time $E(X_m)$ and δ_{REET} (in the brackets) for $(\alpha, \theta) = (2, 0.5)$ under PT-II CBR

(n, m)	p=0.05	p=0.1	p=0.3	p=0.5	p=0.7	p=0.9
15	4.0340	4.0340	4.0340	4.0340	4.0340	4.0340
(15,14)	2.5476	2.8857	3.4785	3.6652	3.7676	3.9679
	(0.6315)	(0.7154)	(0.8623)	(0.9086)	(0.9340)	(0.9836)
(15,13)	1.6483	2.1813	3.2342	3.5110	3.6181	3.7663
	(0.4086)	(0.5407)	(0.8017)	(0.8704)	(0.8969)	(0.9336)
(15,12)	0.7518	1.1819	2.6107	3.3734	3.3846	3.7347
	(0.1864)	(0.2930)	(0.6472)	(0.8362)	(0.8390)	(0.9258)
(15,11)	0.5354	0.8059	2.4429	2.6319	2.8624	3.6419
	(0.1327)	(0.1998)	(0.6056)	(0.6524)	(0.7096)	(0.9028)
(15,10)	0.5104	0.7854	2.2901	2.7439	3.0896	3.6191
	(0.1265)	(0.1947)	(0.5677)	(0.6802)	(0.7659)	(0.8971)
12	3.4955	3.4955	3.4955	3.4955	3.4955	3.4955
(12,11)	1.9313	2.3490	3.1130	3.1289	3.4512	3.4808
	(0.5525)	(0.6720)	(0.8906)	(0.8951)	(0.9873)	(0.9958)
(12,10)	1.2064	1.6751	2.7961	2.8736	2.8852	2.9184
	(0.3451)	(0.4792)	(0.7999)	(0.8221)	(0.8254)	(0.8349)
(12,9)	0.7102	1.1478	2.2872	2.6235	2.7740	3.0725
	(0.2032)	(0.3284)	(0.6543)	(0.7505)	(0.7936)	(0.8790)
(12,8)	0.4466	0.6991	1.8043	2.2604	2.5090	2.5323
	(0.1278)	(0.2000)	(0.5162)	(0.6467)	(0.7178)	(0.7244)
9	2.8353	2.8353	2.8353	2.8353	2.8353	2.8353
(9,8)	1.3739	1.7627	2.4083	2.5943	2.6157	2.7586
	(0.4845)	(0.6217)	(0.8494)	(0.9150)	(0.9225)	(0.9729)
(9,7)	0.7664	1.0396	1.8727	2.2745	2.3097	2.4699
	(0.2703)	(0.3667)	(0.6605)	(0.8022)	(0.8146)	(0.8711)
(9,6)	0.4260	0.5567	1.2967	1.8170	2.0244	2.0305
	(0.1503)	(0.1964)	(0.4573)	(0.6408)	(0.7140)	(0.7161)

TABLE 8. Goodness of fit for the remission times (months) of bladder cancer data

Distribution	AIC	BIC	K-S Statistics	P-value	Log-likelihood
EPD	856.6102	862.3142	0.1016	0.5239	-426.3051
Pareto	948.0433	953.7473	0.3125	7.45E-06	-472.0216

TABLE 9. Bayes and ML estimates, CI and HPD intervals for α with fixed $n = 128$ and $p = 0.3$ under PT-II CBR for the remission times (months) of bladder cancer data for different censoring schemes $(S_{n:m})$.

$S_{n:m}$	MLE	Bayes Estimates(MCMC)			Interval			
		SELF	GELF		95% CI		95% HPD	
			$\delta = 1.5$	$\delta = -1.5$	α_{L^c}	α_{U^c}	α_{L^h}	α_{U^h}
51	3.5109	2.9787	2.9787	2.9787	2.1359	4.8859	2.9635	2.9925
64	2.9321	2.9273	2.9273	2.9273	1.8982	3.9659	2.9141	2.9398
77	3.5733	3.5675	3.5674	3.5675	2.3763	4.7704	3.5528	3.5829
90	3.6080	3.6033	3.6033	3.6033	2.4537	4.7624	3.5882	3.6168
102	4.0606	4.0564	4.0563	4.0564	2.8117	5.3096	4.0416	4.0724
128	4.6574	4.6574	4.6573	4.6329	3.3135	6.0013	4.6409	4.6740

TABLE 10. Bayes and ML estimates, CI and HPD intervals for θ with fixed $n = 128$ and $p = 0.3$ under PT-II CBR for the remission times (months) of bladder cancer data for different censoring schemes $(S_{n:m})$.

$S_{n:m}$	MLE	Bayes Estimates(MCMC)			Interval			
		SELF	GELF		95% CI		95% HPD	
			$\delta = 1.5$	$\delta = -1.5$	θ_{L^c}	θ_{U^c}	θ_{L^h}	θ_{U^h}
51	0.6885	0.6895	0.6895	0.6895	0.4883	0.8885	0.68512	0.6937
64	0.7123	0.7112	0.7112	0.7112	0.5195	0.9051	0.7076	0.7152
77	0.8203	0.8193	0.8193	0.8193	0.6373	1.0034	0.8157	0.8227
90	0.8824	0.8816	0.8816	0.8816	0.7023	1.0624	0.8779	0.8849
102	0.9559	0.9554	0.9554	0.9554	0.7823	1.1296	0.9519	0.9586
128	1.0877	1.0877	1.0877	1.0845	0.9194	1.2559	1.0845	1.0912

TABLE 11. PT-II CBR under different censoring schemes $S_{n:m}$ for fixed $n = 128$ and $p = 0.3$ for the remission times (months) of bladder cancer data

$S_{m:n}$	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$S_{51:128}$	y_i	0.08	0.2	0.4	0.5	0.81	0.9	1.05	1.26	1.35	1.4	1.46	1.76	2.02	2.02	2.09	2.23	2.46
	R_i	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	1	0
	y_i	2.54	2.62	2.64	2.69	2.75	2.83	2.87	3.02	3.31	3.36	3.36	3.48	3.57	3.64	3.7	3.88	4.18
	R_i	0	0	0	1	0	0	0	1	0	0	0	1	0	0	1	0	0
	y_i	4.23	4.26	4.33	4.4	4.5	4.51	4.87	4.98	5.06	5.09	5.17	5.32	5.32	5.34	5.41	5.49	5.62
	R_i	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	67
$S_{64:128}$	y_i	0.08	0.2	0.4	0.5	0.51	0.81	0.9	1.05	1.19	1.26	1.35	1.4	1.46	1.76	2.02	2.02	2.07
	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	y_i	2.09	2.23	2.26	2.46	2.54	2.62	2.64	2.69	2.75	3.02	3.31	3.36	3.48	3.52	3.57	3.64	3.7
	R_i	0	0	0	0	0	0	0	1	2	1	1	0	0	0	0	0	0
	y_i	3.82	3.88	4.18	4.26	4.33	4.34	4.4	4.5	4.51	4.87	5.06	5.09	5.32	5.32	5.34	5.41	5.49
	R_i	0	0	1	0	0	0	0	0	0	1	0	1	0	0	0	1	0
	y_i	5.62	5.71	5.85	6.25	6.54	6.76	6.93	7.09	7.26	7.28	7.32	7.39	7.62				
	R_i	0	0	0	0	0	0	2	0	0	0	0	1	52				
$S_{77:128}$	y	0.08	0.2	0.4	0.5	0.81	0.9	1.05	1.26	1.35	1.4	1.46	1.76	2.02	2.02	2.07	2.09	2.23
	R_i	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	1
	y_i	2.46	2.62	2.64	2.69	2.83	2.87	3.02	3.25	3.31	3.36	3.36	3.52	3.64	3.7	3.82	3.88	4.18
	R_i	1	0	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0
	y_i	4.23	4.26	4.33	4.5	4.51	4.87	4.98	5.06	5.09	5.17	5.32	5.34	5.41	5.41	5.49	5.62	6.25
	R_i	0	0	2	0	0	0	0	0	0	1	0	0	0	0	0	2	1
	y_i	6.76	6.93	6.94	6.97	7.09	7.26	7.28	7.32	7.39	7.59	7.62	7.63	7.66	7.87	7.93	8.26	8.37
	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	y_i	8.65	8.66	9.02	9.47	9.74	10.06	10.34	10.66	10.75								
	R_i	0	0	1	0	0	0	0	0	35								
$S_{90:128}$	y_i	0.08	0.2	0.4	0.5	0.51	0.81	0.9	1.05	1.19	1.26	1.35	1.4	1.46	1.76	2.02	2.02	2.07
	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	y_i	2.23	2.26	2.46	2.54	2.62	2.64	2.69	2.69	2.75	2.83	2.87	3.02	3.25	3.31	3.36	3.36	3.48
	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
	y_i	3.57	3.64	3.7	3.82	3.88	4.18	4.23	4.26	4.33	4.34	4.4	4.5	4.51	4.87	4.98	5.06	5.09
	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	y_i	5.17	5.32	5.32	5.34	5.41	5.41	5.62	5.71	5.85	6.25	6.54	6.76	6.93	6.94	6.97	7.09	7.26
	R_i	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
	y_i	7.28	7.32	7.39	7.59	7.62	7.63	7.87	7.93	8.26	8.37	8.53	8.65	8.66	9.02	9.22	9.74	10.06
	R_i	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0
	y_i	10.34	10.66	11.64	11.79	11.98												
	R_i	0	2	0	0	31												
$S_{102:128}$	y_i	0.08	0.2	0.4	0.5	0.51	0.81	0.9	1.19	1.26	1.35	1.4	1.46	1.76	2.02	2.07	2.09	2.23
	R_i	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0
	y_i	2.26	2.46	2.54	2.62	2.64	2.69	2.69	2.75	2.87	3.02	3.25	3.31	3.36	3.36	3.48	3.52	3.57
	R_i	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1
	y_i	3.7	3.82	3.88	4.18	4.23	4.26	4.33	4.34	4.4	4.5	4.51	4.87	4.98	5.06	5.09	5.17	5.32
	R_i	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	y_i	5.32	5.34	5.41	5.41	5.49	5.62	5.71	5.85	6.25	6.76	6.93	6.94	6.97	7.09	7.28	7.32	7.39
	R_i	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0
	y_i	7.59	7.63	7.66	7.87	7.93	8.26	8.37	8.53	8.65	8.66	9.02	9.22	9.47	9.74	10.06	10.34	10.66
	R_i	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	y_i	10.75	11.25	11.64	11.79	11.98	12.02	12.07	12.63	13.11	13.29	13.8	14.24	14.76	14.77	14.83	15.96	16.62
	R_i	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	18