STATISTICS IN TRANSITION-new series, July 2009
Vol. 10, No. 1, pp. 1

SUBMISSION INFORMATION FOR AUTHORS

Statistics in Transition – new series (SiT) is an international journal published jointly by the Polish Statistical Association (PTS) and the Central Statistical Office of Poland, on a quarterly basis (during 1993—2006 it was issued twice and since 2006 three times a year). Also, it has extended its scope of interest beyond its originally primary focus on statistical issues pertinent to transition from centrally planned to a market-oriented economy through embracing questions related to systemic transformations of and within the national statistical systems, world-wide.

The SiT-*ns* seeks contributors that address the full range of problems involved in data production, data dissemination and utilization, providing international community of statisticians and users — including researchers, teachers, policy makers and the general public — with a platform for exchange of ideas and for sharing best practices in all areas of the development of statistics.

Accordingly, articles dealing with any topics of statistics and its advancement — as either a scientific domain (new research and data analysis methods) or as a domain of informational infrastructure of the economy, society and the state — are appropriate for *Statistics in Transition new series.*

Demonstration of the role played by statistical research and data in economic growth and social progress (both locally and globally), including better-informed decisions and greater participation of citizens, are of particular interest.

Each paper submitted by prospective authors are peer reviewed by internationally recognized experts, who are guided in their decisions about the publication by criteria of originality and overall quality, including its content and form, and of potential interest to readers (esp. professionals).

Manuscript should be submitted electronically to the Editor:
sit@stat.gov.pl., followed by a hard copy addressed to
Prof. Wlodzimierz Okrasa,
GUS / Central Statistical Office
Al. Niepodległości 208, R. 287, 00-925 Warsaw, Poland

It is assumed, that the submitted manuscript has not been published previously and that it is not under review elsewhere. It should include an abstract (of not more than 1600 characters, including spaces). Inquiries concerning the submitted manuscript, its current status etc., should be directed to the Editor by email, address above, or w.okrasa@stat.gov.pl.

For other aspects of editorial policies and procedures see the SiT Guidelines on its Web site: http://www.stat.gov.pl/gus/5840_2638_ENG_HTML.htm

STATISTICS IN TRANSITION-new series, July 2009
Vol. 10, No. 1, pp. 3—14

ON CLASS OF ESTIMATORS FOR POPULATION MEAN USING MULTI-AUXILIARY CHARACTERS IN THE PRESENCE OF NON-RESPONSE

B. B. Khare[1] and R. R. Sinha

ABSTRACT

Two classes of estimators for the population mean of the study character using multi-auxiliary characters with known population means in presence of non-response have been proposed. The expressions for bias, mean square error and conditions for attaining minimum mean square error of the proposed classes of estimators have been obtained. An empirical study has also been given in support of the problem.

Key words: Population Mean; Bias; Mean square error; Non-response.

1. Introduction

The problem of estimating the population mean using multi-auxiliary characters with known population means has been considered by several authors [Olkin (1958), Shukla (1965,66), Rao and Mudholkar (1967), Mohanty (1967,70), Srivastava (1971), Khare and Srivastava (1980,81), Khare (1983), Srivastava and Jhaji (1983), Sahoo (1986), Sampath (1989)] when there is no non-response on sample units. However, it is a common experience that in sample surveys, the data may not be collected for all the units selected in the sample due to the problem of non-response. Frequently it has been observed in demographic surveys that some units selected in the sample may not respond on the study character (y) while the data on all the units are available for auxiliary characters (x). If there is non-response on some sample units then the estimators for population mean using auxiliary character in presence of non-response have been considered by Rao (1986, 90), Khare (2002) and Khare and Srivastava (1996, 97, 2000, 2002).

In the present context we have proposed two classes of estimators t_1 and t_1^* for the population mean of the study character using multi-auxiliary characters

[1] Department of Statistics, Banaras Hindu University, Varanasi, INDIA, bbkhare56@yahoo.com.

with known population means in the presence of non-response and studied their properties. The expressions for bias and mean square error of the proposed classes of estimators have been obtained. The conditions for minimising the mean square error of the proposed classes of estimators have also been obtained. The efficiency of the proposed classes of estimators with respect to the relevant estimator has been shown with the help of an empirical study.

2. The estimators

Let y and x_j (j = 1,2,...,p) be the main and auxiliary characters under study having non-negative ℓ-th value (Y_ℓ, $X_{j\ell}$; ℓ = 1,2,...,N and j = 1,2,...,p) with population mean $\overline{Y}$ and $\overline{X}_j$ (j = 1,2,...,p) respectively. The whole population is supposed to be divided in N_1 responding and N_2 non-responding units such that $N_1 + N_2 = N$. A sample of size n is drawn by using SRSWOR method of sampling from the population of N and it is observed that out of these n selected units, n_1 units respond and n_2 units do not respond. We have considered that the responding and non-responding units are same for study character as well as auxiliary characters. Further, from n_2 non-responding units, a subsample of size 'r' ($r = n_2/k$; $k > 1$) is drawn by making extra effort using SRSWOR technique of sub-sampling and consequently the estimator $\overline{y}^*$ [Hansen and Hurwitz (1946)] for the population mean $\overline{Y}$ is given by

$$\overline{y}^* = \frac{n_1}{n}\overline{y}_1 + \frac{n_2}{n}\overline{y}_2' \tag{2.1}$$

where $\overline{y}_1$ and $\overline{y}_2'$ are the sample means of characters y based on n_1 and r units respectively. The estimator $\overline{y}^*$ is unbiased and the variance of $\overline{y}^*$ is given by

$$V(\overline{y}^*) = \frac{1-f}{n} S_0^2 + \frac{W_2(k-1)}{n} S_{0(2)}^2 , \tag{2.2}$$

where $W_i = \frac{N_i}{N}$ (i = 1,2), $f = \frac{n}{N}$ and S_0^2, $S_{0(2)}^2$ denote the population mean square of y for the entire population and for the non-responding part of the population.

Similarly, the estimator $\overline{x}_j^*$ (j = 1,2,...,p) for the population means $\overline{X}_j$ (j = 1,2,...,p) is given by

$$\bar{x}_j^* = \frac{n_1}{n}\bar{x}_{j(1)} + \frac{n_2}{n}\bar{x}'_{j(2)}\,, \tag{2.3}$$

where $\bar{x}_{j(1)}$ and $\bar{x}'_{j(2)}$ are the sample means of character x_j based on n_1 and r units respectively. The variance of $\bar{x}_j^*$ is given by

$$V(\bar{x}_j^*) = \frac{1-f}{n}S_j^2 + \frac{W_2(k-1)}{n}S_{j(2)}^2\,, \tag{2.4}$$

where S_j^2 and $S_{j(2)}^2$ denote the population mean square of x_j for the entire population and for the non-responding part of the population respectively.

Now, we proposed the classes of estimators for population mean by utilizing the multi-auxiliary characters with known population means under two different cases which are given as follows:

Case I: Incomplete Information on y and x_j (j = 1,2,...,p)

When population means of x_1, x_2,..., x_p are known and only n_1 of the n units respond on both y and x_j (j = 1,2,...,p), then we propose a class of estimator t_1 for population mean $\bar{Y}$ using multi-auxiliary characters with known population means in presence of non-response which is as follows:

$$t_1 = g(v, \mathbf{u}')\,, \tag{2.5}$$

such that

$$g(\bar{Y}, \mathbf{e}') = \bar{Y}\,, \quad g_1(\bar{Y}, \mathbf{e}') = \left(\frac{\partial}{\partial v}g(v, \mathbf{u}')\right)_{(\bar{Y}, \mathbf{e}')} = 1\,, \tag{2.6}$$

where **u** and **e** denote the column vectors $(u_1,u_2,...,u_p)'$ and $(1,1,...,1)'$ respectively. We also denote $v = \bar{y}^*$ and $u_j = \bar{x}_j^* / \bar{X}_j \quad (j = 1,2,\ldots,p)$.

Case II: Complete Information on x_j (j = 1, 2,...,p) But Incomplete Information on y

When population means of $x_1,x_2,...,x_p$ are known and only n_1 of the n units responds on y but there is complete information on x_j ; j = 1,2,..,p for all the n units of the sample, then we propose a class of estimators t_1^* for population mean $\bar{Y}$ using multi-auxiliary characters with known population means in presence of non-response on y only which is given as follows:

$$t_1^* = h(v, \boldsymbol{\omega}')\,, \tag{2.7}$$

such that

$$h(\overline{Y},\mathbf{e}')=\overline{Y}\,,\quad h_1(\overline{Y},\mathbf{e}')=\left(\frac{\partial}{\partial v}h(v,\boldsymbol{\omega}')\right)_{(\overline{Y},\mathbf{e}')}=1\,, \qquad (2.8)$$

where $\boldsymbol{\omega}$ denotes the column vector $(\omega_1,\omega_2,...,\omega_p)$ and

$$\omega_j=\overline{x}_j/\overline{X}_j \quad (j = 1, 2,...,p)$$

The functions $g(v,\mathbf{u}')$ and $h(v,\boldsymbol{\omega}')$ also satisfy the following conditions -

(i) For any sampling design, whatever be the sample chosen, $(v,\mathbf{u}')$ and $(v,\boldsymbol{\omega}')$ assume values in a bounded closed convex subset D_1 and D_2 respectively of p+1 dimensional real space containing the point $(\overline{Y},\mathbf{e}')$.

(ii) The functions $g(v,\mathbf{u}')$ and $h(v,\boldsymbol{\omega}')$ and their first and second partial derivatives exist and are continuous and bounded in D_1 and D_2 respectively.

Here $[g_1(v,\mathbf{u}'),\ g_2(v,\mathbf{u}')]$ and $[h_1(v,\boldsymbol{\omega}'),\ h_2(v,\boldsymbol{\omega}')]$ denote the first partial derivatives of $g(v,\mathbf{u}')$ and $h(v,\boldsymbol{\omega}')$ with respect to $[v,\ \mathbf{u}']$ and $[v,\ \boldsymbol{\omega}']$ respectively. The second partial derivative of $g(v,\mathbf{u}')$, $h(v,\boldsymbol{\omega}')$ with respect to $\mathbf{u}'$ and $\boldsymbol{\omega}'$ are denoted by $g_{22}(v,\mathbf{u}')$, $h_{22}(v,\boldsymbol{\omega}')$ and first partial derivative of $g_2(v,\mathbf{u}')$ and $h_2(v,\boldsymbol{\omega}')$ with respect to v are denoted by $g_{12}(v,\mathbf{u}')$ and $h_{12}(v,\boldsymbol{\omega}')$.

On occurrence of the regularity conditions imposed on $g(v,\mathbf{u}')$ and $h(v,\boldsymbol{\omega}')$, it may be seen that the bias and mean square error of the estimators t_1 and t_1^* will always exist.

3. Bias and mean square error (MSE)

Expand the functions $g(v,\mathbf{u}')$ and $h(v,\boldsymbol{\omega}')$ about the point $(\overline{Y},\mathbf{e}')$ using Taylor's series up to second order partial derivatives, under the conditions (2.6) and (2.8), the expressions for bias and mean square error of the proposed classes of estimators t_1 and t_1^* for any sampling design up to the terms of order n^{-1} are given by

$$\text{Bias}\,(t_1)=E(v-\overline{Y})\,(\mathbf{u}-\mathbf{e})'\ g_{12}(v^*,\mathbf{u}^{*\prime})+\frac{1}{2}E(\mathbf{u}-\mathbf{e})'\,g_{22}(v^*,\mathbf{u}^{*\prime})\,(\mathbf{u}-\mathbf{e})\,, \qquad (3.1)$$

$$\text{MSE}\,(t_1)=V(\overline{y}^*)+2E(v-\overline{Y})\,(\mathbf{u}-\mathbf{e})'\ g_2(\overline{Y},\mathbf{e}')$$

$$+\ E(g_2(\overline{Y},\mathbf{e}'))'\ (\mathbf{u}-\mathbf{e})\ (\mathbf{u}-\mathbf{e})'\ g_2(\overline{Y},\mathbf{e}')\ , \tag{3.2}$$

$$\text{Bias}\ (t_1^*) = E(v-\overline{Y})\ (\omega-\mathbf{e})'\ h_{12}(v^*,\omega^{*\prime}) + \frac{1}{2}E(\omega-\mathbf{e})'\ h_{22}(v^*,\omega^{*\prime})\ (\omega-\mathbf{e}) \tag{3.3}$$

and

$$\text{MSE}\,(t_1^*) = V(\bar{y}^*) + 2E(v-\overline{Y})\ (\boldsymbol{\omega}-\mathbf{e})'\ h_2(\overline{Y},\mathbf{e}')$$

$$+\ E(h_2(\overline{Y},\mathbf{e}'))'(\boldsymbol{\omega}-\mathbf{e})\ (\boldsymbol{\omega}-\mathbf{e})'\ h_2(\overline{Y},\mathbf{e}')\ , \tag{3.4}$$

where $v^* = \overline{Y} + \phi\,(v-\overline{Y})$, $\mathbf{u}^* = \mathbf{e} + \boldsymbol{\phi}_1(\mathbf{u}-\mathbf{e})$, $\boldsymbol{\omega}^* = \mathbf{e}+\boldsymbol{\phi}_2(\boldsymbol{\omega}-\mathbf{e})$, while $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ denote the diagonal matrix of order $p \times p$ having j^{th} diagonal elements ϕ_{1j} and ϕ_{2j} $(j = 1,2,...,p)$ respectively, such that

$$0 < \phi,\ \phi_{1j},\ \phi_{2j} < 1\ , \quad \forall\, j = 1,2,...,p.$$

The mean square error of t_1 and t_1^* will attain minimum value for

$$g_2(\overline{Y},\mathbf{e}') = -(E(\mathbf{u}-\mathbf{e})\ (\mathbf{u}-\mathbf{e})')^{-1}\,.\,E(v-\overline{Y})\ (\mathbf{u}-\mathbf{e}) \tag{3.5}$$

and

$$h_2(\overline{Y},\mathbf{e}') = -(E(\boldsymbol{\omega}-\mathbf{e})\ (\boldsymbol{\omega}-\mathbf{e})')^{-1}\,.\,E(v-\overline{Y})\ (\boldsymbol{\omega}-\mathbf{e}) \tag{3.6}$$

respectively. The minimum values of MSE(t) and $\text{MSE}\,(t_1^*)$ are given by

$$\text{MSE}(t_1)_{\text{min.}} = V(\bar{y}^*) - E(v-\overline{Y})\ (\mathbf{u}-\mathbf{e})'\ (E(\mathbf{u}-\mathbf{e})\ (\mathbf{u}-\mathbf{e})')^{-1}\ E(v-\overline{Y})\ (\mathbf{u}-\mathbf{e}) \tag{3.7}$$

and

$$\text{MSE}(t_1^*)_{\text{min.}} = V(\bar{y}^*) - E(v-\overline{Y})\ (\boldsymbol{\omega}-\mathbf{e})'\ (E(\boldsymbol{\omega}-\mathbf{e})\ (\boldsymbol{\omega}-\mathbf{e})')^{-1}\ E(v-Y)\ (\boldsymbol{\omega}-\mathbf{e}) \tag{3.8}$$

Considering SRSWOR method of sampling, let $\mathbf{A} = [a_{jj'}]$ and $\mathbf{A}_0 = [a_{0jj'}]$ are two $p \times p$ positive definite matrix such that

$$a_{jj'} = \frac{1-f}{n}\rho_{jj'}\ C_j\ C_{j'} + \frac{W_2(k-1)}{n}\rho_{jj'(2)}\ C'_j\ C'_{j'}$$

and

$$a_{0jj'} = \rho_{jj'}\ C_j\ C_{j'} \qquad\qquad \forall\ j \neq j' = 1,2,...,p$$

Also let $\mathbf{b} = (b_1, b_2 ..., b_p)'$ and $\mathbf{d} = (d_1, d_2 ..., d_p)'$ be two column vectors such that

$$b_j = \rho_j C_j \quad \text{and} \quad d_j = \rho_{j(2)} C'_j ,$$

where $C_j^2 = S_j^2 / \overline{X}_j^2$, $C_j'^2 = S_{j(2)}^2 / \overline{X}_j^2$.

Here, we define $\rho_{jj'}$ and $\rho_{jj'(2)}$ to be the correlation coefficients between x_j and $x_{j'}$ for the entire population and for the non-responding group of the population respectively while ρ_j and $\rho_{j(2)}$ to be the correlation coefficients between y and x_j for the entire and non-responding group of the population respectively.

Now, in the case of SRSWOR methods of sampling, the expressions for bias and mean square error of t_1 and t_1^* up to the terms of order (n^{-1}) are given as follows:

$$\text{Bias}(t_1) = \left(\frac{1-f}{n} S_0\, \mathbf{b} + \frac{W_2(k-1)}{n} S_{0(2)}\, \mathbf{d}\right)' g_{12}(v^*, \mathbf{u}^{*\prime})$$

$$+ \frac{1}{2} \text{trace}\, \mathbf{A}\;\; g_{22}(v^*, \mathbf{u}^{*\prime}) , \tag{3.9}$$

$$\text{MSE}(t_1) = V(\overline{y}^*) + (g_2(\overline{Y}, \mathbf{e}'))' \mathbf{A}\, g_2(\overline{Y}, \mathbf{e}')$$

$$+2\left(\frac{1-f}{n} S_0\, \mathbf{b} + \frac{W_2(k-1)}{n} S_{0(2)}\, \mathbf{d}\right)' g_2(\overline{Y}, \mathbf{e}') , \tag{3.10}$$

$$\text{Bias}(t_1^*) = \frac{1-f}{n}\left(S_0\, \mathbf{b}'\;\; h_{12}(v^*, \boldsymbol{\omega}^{*\prime}) + \frac{1}{2} \text{trace}\, \mathbf{A}_0\;\; h_{22}(v^*, \boldsymbol{\omega}^{*\prime})\right), \tag{3.11}$$

and

$$\text{MSE}(t_1^*) = V(\overline{y}^*) + \frac{1-f}{n}[(h_2(\overline{Y}, \mathbf{e}'))' \mathbf{A}_0\, h_2(\overline{Y}, \mathbf{e}') + 2S_0\, \mathbf{b}'\, h_2(\overline{Y}, \mathbf{e}')] \tag{3.12}$$

The values of $g_2(\overline{Y}, \mathbf{e}')$ and $h_2(\overline{Y}, \mathbf{e}')$ for which MSE(t_1) and MSE (t_1^*) will attain minimum value respectively are given by

$$g_2(\overline{Y}, \mathbf{e}') = -\mathbf{A}^{-1}\left(\frac{1-f}{n} S_0\, \mathbf{b} + \frac{W_2(k-1)}{n} S_{0(2)}\, \mathbf{d}\right) \tag{3.13}$$

and

$$h_2(\overline{Y}, \mathbf{e}') = - S_0 \mathbf{A}_0^{-1} \mathbf{b} . \quad (3.14)$$

The values of minimum mean square error for the estimators t_1 and t_1^* are given by

$$MSE(t_1)_{min.} = V(\overline{y}^*) - \left(\frac{1-f}{n} S_0 \mathbf{b} + \frac{W_2(k-1)}{n} S_{0(2)} \mathbf{d} \right)'$$

$$\mathbf{A}^{-1} \left(\frac{1-f}{n} S_0 \mathbf{b} + \frac{W_2(k-1)}{n} S_{0(2)} \mathbf{d} \right) \quad (3.15)$$

and

$$MSE(t_1^*)_{min.} = V(\overline{y}^*) - \left(\frac{1-f}{n} \right) S_0^2 \mathbf{b}' \mathbf{A}_0^{-1} \mathbf{b} . \quad (3.16)$$

4. Some particular members of the proposed class of estimators t_1 and t_1^*

The proposed classes of estimators t_1 and t_1^* are the wider class of estimators. All the members belonging to the classes of estimators t_1 and t_1^* attain minimum value of mean square error given in (3.15) and (3.16) if the conditions (3.13) and (3.14) are satisfied respectively. Now, we consider the members of the classes of estimators t_1 and t_1^* which are as follows:

$$T_{01} = v \exp\left[\sum_{j=1}^{p} \theta_{1j} \log u_j \right], \quad (4.1)$$

$$T_{02} = v \sum_{j=1}^{p} W_j u_j^{\theta_{2j} / W_j}, \qquad \sum_{i=1}^{p} W_j = 1. \quad (4.2)$$

$$T_{03} = \sum_{j=1}^{p} [W_j u_j^{\theta_{3j} / W_j}] [v + \beta_{1j}(u_j - 1)]. \quad (4.3)$$

$$T_{01}^{*} = v \exp\left[\sum_{j=1}^{p} \alpha_{1j} \log \omega_j\right], \tag{4.4}$$

$$T_{02}^{*} = v \sum_{j=1}^{p} W_j\, \omega_j^{\alpha_{2j}/W_j}, \qquad \sum_{i=1}^{p} W_j = 1. \tag{4.5}$$

and

$$T_{03}^{*} = \sum_{j=1}^{p} [W_j\, \omega_j^{\alpha_{3j}/W_j}]\, [v + \beta_{2j}(\omega_j - 1)] \tag{4.6}$$

The minimum mean square value of the classes of estimators t_1 and t_1^{*} can be obtained by putting the optimum value of the constants involved in the mean square error of the estimators which are the members of the proposed classes. Sometimes the members of the class attain minimum mean square error for some specified conditions which may or may not exist. In such cases these estimators are rarely used, when the specified conditions are not satisfied. Further, the proposed classes of estimators t_1 and t_1^{*} attain minimum value of mean square error if the conditions (3.13) and (3.14) are respectively satisfied. These conditions involve the unknown parameters, so for obtaining the required value of the constants, one may use the value of the parameters based on past data or experience. In case when past data is not available, one may estimate the required parameters based on preliminary sample. Reddy (1978) has shown that such values are stable overtime and region and do not affect the variance of the estimators up to the terms of order (n^{-1}) [Srivastava and Jhajj (1983)]. Therefore, the use of proposed classes of estimators is recommended in large scale sample surveys.

5. An empirical study

96 village wise population of rural area under Police-station – Singur, District – Hooghly, West Bengal has been taken under the study from the District Census Handbook 1981. The 25% villages (i.e. 24 villages) whose area is greater than 160 hectares have been considered as non-response group of the population. The number of agricultural labours in the village is taken as study character (y) while the area (in hectares) of the village, the number of cultivators in the village and the total population of the village are taken as auxiliary characters x_1, x_2 and x_3 respectively. The values of the parameters of the population under study are as follows:

$\overline{Y} = 137.9271$	$\overline{X}_1 = 144.8720$	$\overline{X}_2 = 185.2188$	$\overline{X}_3 = 1807.8958$
$S_0 = 182.5012$	$C_1 = 0.8115$	$C_2 = 1.0529$	$C_3 = 1.0591$
$S_{0(2)} = 287.4204$	$C'_1 = 0.9408$	$C'_2 = 1.4876$	$C'_3 = 1.5331$
$\rho_1 = 0.773$	$\rho_{1(2)} = 0.724$	$\rho_{12} = 0.819$	$\rho_{12(2)} = 0.724$
$\rho_2 = 0.786$	$\rho_{2(2)} = 0.787$	$\rho_{13} = 0.877$	$\rho_{13(2)} = 0.853$
$\rho_3 = 0.904$	$\rho_{3(2)} = 0.895$	$\rho_{23} = 0.906$	$\rho_{23(2)} = 0.891$

To study the performance of the proposed classes of estimators t_1 and t_1^* with respect to $\overline{y}*$, we consider the estimators

$$T_{01} = v \exp\left[\sum_{j=1}^{p} \theta_{1j} \log u_j\right],$$

and

$$T_{01}^* = v \exp\left[\sum_{j=1}^{p} \alpha_{1j} \log \omega_j\right],$$

which are the members of the proposed classes of estimators t_1 and t_1^* respectively. The optimum values of the constant θ_{1j} and α_{1j} involved in T_{01} and T_{01}^* can be respectively calculated by the equation (3.13) and (3.14), which are as follows:

$$T_{01}\begin{cases} \text{For } k=4 \begin{cases} T_{01}\ (p=1):\ \theta_{11} = -1.4834 \\ T_{01}\ (p=2):\ \theta_{11} = -0.6614,\ \theta_{12} = -0.7372 \\ T_{01}\ (p=3):\ \theta_{11} = 0.2897,\ \theta_{12} = 0.1208,\ \theta_{13} = -1.4602 \end{cases} \\ \text{For } k=3 \begin{cases} T_{01}\ (p=1):\ \theta_{11} = -1.4487 \\ T_{01}\ (p=2):\ \theta_{11} = -0.6481,\ \theta_{12} = -0.7234 \\ T_{01}\ (p=3):\ \theta_{11} = 0.2684,\ \theta_{12} = 0.1253,\ \theta_{13} = -1.4481 \end{cases} \\ \text{For } k=2 \begin{cases} T_{01}\ (p=1):\ \theta_{11} = -1.3889 \\ T_{01}\ (p=2):\ \theta_{11} = -0.6316,\ \theta_{12} = -0.6931 \\ T_{01}\ (p=3):\ \theta_{11} = 0.2271,\ \theta_{12} = 0.1363,\ \theta_{13} = -1.4252 \end{cases} \end{cases}$$

and

$$T_{01}^{*} \begin{cases} T_{01}^{*}\ (p=1) : \alpha_{11} = -1.2606 \\ T_{01}^{*}\ (p=2) : \alpha_{11} = -0.6447,\ \alpha_{12} = -0.5796 \\ T_{01}^{*}\ (p=3) : \alpha_{11} = 0.1013,\ \alpha_{12} = 0.1838,\ \alpha_{13} = -1.3589 \end{cases}$$

The mean square error and relative efficiency of t_1 and t_1^* with respect to $\bar{y}^*$ for one, two and three auxiliary characters with different values of the subsampling fraction (1/k) are given in Table 1.

Table 1. R.E.(·) in % with respect to $\bar{y}^*$ for different values of k

Estimators	Auxiliary character(s)	N = 96, n = 40					
		1/k					
		1/4		1/3		1/2	
$\bar{y}^*$	–	100.00	(2034.5580)*	100.00	(1518.2425)	100.00	(1001.9270)
T_{01}	x_1	215.97	(942.0470)	218.23	(695.7218)	223.76	(447.7739)
	x_1, x_2	298.33	(681.9845)	297.17	(510.8925)	296.09	(338.3828)
	x_1, x_2, x_3	525.21	(387.3806)	527.35	(287.8993)	532.49	(188.1597)
T_{01}^*	x_1	116.64	(1744.3606)	123.63	(1228.0451)	140.77	(711.7296)
	x_1, x_2	118.97	(1710.0916)	127.18	(1193.7761)	147.90	(667.4606)
	x_1, x_2, x_3	124.35	(1636.1917)	135.57	(1119.8762)	166.06	(603.3607)

**Figures in parenthesis represent MSE(·).*

From the Table 1, it is very clear that the mean square error of T_{01} and T_{01}^* decreases as the numbers of auxiliary characters and subsampling fractions increase. The estimators T_{01} and T_{01}^* are more efficient than $\bar{y}^*$ for different values of k while comparing T_{01} with T_{01}^* it is observed that the estimator T_{01} is more efficient than T_{01}^* and their efficiencies are increasing with respect to $\bar{y}^*$ when numbers of auxiliary characters and subsampling fractions increase. Hence we conclude that the efficiency of all the members of the class of estimators t_1 or t_1^* with respect to $\bar{y}^*$ can be increased by increasing the numbers of auxiliary characters as well as by increasing the value of subsampling fractions to be selected from non-responding group.

REFERENCES

HANSEN, M.H. and HURWITZ, W.N. (1946): The problem of nonresponse in sample surveys. Jour. Amer. Statist. Assoc., 41, 517–529.

KHARE, B.B. (1983): Some problems of estimation using auxiliary character. Ph.D. Thesis submitted to B.H.U., Varanasi, India.

KHARE, B.B. (2002): A class of estimators for population mean using auxiliary character in presence of soft core observations. Prog. of Maths., 36 (1 & 2), 1–9.

KHARE, B.B. and SRIVASTAVA, S.R. (1980): On an efficient estimator of population mean using two auxiliary variables. Proc. Nat. Acad. Scie., India, 50(A), 209–214.

KHARE, B.B. and SRIVASTAVA, S.R. (1981): A generalized regression ratio estimator for the population mean using two auxiliary variables. Aligarh Jour. Statist., 1(1), 43–51.

KHARE, B.B. and SRIVASTAVA, S. (1996): Transformed product type estimators for population mean in presence of softcore observations. Proc. Math. Soc. BHU, 12, 29–34.

KHARE, B.B. and SRIVASTAVA, S. (1997): Transformed ratio type estimators for the population mean in the presence of non-response. Commun. Statis.-Theory. Math., 26(7), 1779–1791.

KHARE, B.B. and SRIVASTAVA, S. (2000): Generalised estimators for population mean in presence of nonresponse, Internal. J. Math. & Statist. Sci. 9(1), 75–87.

KHARE, B.B. and SRIVASTAVA, S. (2002): Study of two phase sampling ratio type estimators for population mean in presence of non-response under super-population model approach. Proc. Nat. Acad. Sci. India, 72 (A) IV, 331–337.

MOHANTI, S. (1967): Combination of regression and ratio estimate. Jour. Ind. Stat. Assoc., 5, 16–29.

MOHANTY, S. (1970): Contribution to the theory of sampling. Ph.D. thesis submitted to I.A.S.R.I., New Delhi.

OLKIN, I. (1958): Multivariate ratio estimation for finite populations. Biometrika, 45, 154–165.

RAO, P.S.R.S. (1986): Ratio estimation with subsampling the nonrespondents. Survey Methodology, 12(2), 217–230.

RAO, P.S.R.S. (1990): Regression estimators with subsampling of nonrespondents. In-Data Quality Control, Theory and Pragmatics, (Eds.) Gunar E. Liepins and V.R.R. Uppuluri, Marcel Dekker, New York, (1990), pp. 191–208.

RAO, P.S.R.S. and MUDHOLKAR, G.S. (1967): Generalized multivariate estimators for the mean of a finite population. Jour. Amer. Statist. Assoc., 62, 1009–1012.

REDDY, V.N. (1978): A study of use of prior knowledge on certain population parameters in estimation. *Sankhya*, C, **40**, 29–37.

SAHOO, L.N. (1986): On a class of unbiased estimators using multi-auxiliary information. Jour. Ind. Soc. Agri. Statist., 38, 379–382.

SAMPATH, S. (1989): On the optimal choice of unknowns in Ratio-Type estimators. J. Indian Soc. Agricultural Statist., 41, 166–172.

SHUKLA, G.K. (1965): Multivariate regression estimate. Jour. Ind. Stat. Assoc., 3, 202–211.

SHUKLA, G.K. (1966): An alternate multivariate ratio estimate for finite population. Cal. Stat. Assoc. Bull., 15, 127–134.

SRIVASTAVA, S.K. (1971): A generalized estimator for the mean of a finite population using multiauxiliary information. Jour. Amer. Statist. Assoc., 66, 404–407.

SRIVASTAVA, S.K, and JHAJJ, H.S. (1983): A class of estimators of the population mean using multi-auxiliary information. Cal. Stat. Assoc. Bull., 32, 47–56.

STATISTICS IN TRANSITION-new series, July 2009
Vol. 10, No. 1, pp. 15—24

A MODIFIED RATIO-CUM-PRODUCT ESTIMATOR OF FINITE POPULATION MEAN USING KNOWN COEFFICIENT OF VARIATION AND COEFFICIENT OF KURTOSIS

Rajesh Tailor[1] and Balkishan Sharma

ABSTRACT

This paper proposes a ratio-cum-product estimator of finite population mean using information on coefficient of variation and coefficient of kurtosis of auxiliary variate. The bias and mean squared error of the proposed estimator are obtained. It has been shown that the proposed estimator is more efficient than the sample mean estimator, usual ratio and product estimators and estimators proposed by Upadhyaya and Singh (1999) under certain given conditions. An empirical study is also carried out to demonstrate the merits of the proposed estimator over other estimators.

Key words: Coefficient of Variation, Coefficient of Kurtosis, Finite Population Mean, Bias and Mean Squared Error.

1. Introduction

In sample surveys, auxiliary information is used at both selection as well as estimation stages to improve the efficiency of the estimators. When the correlation between study variate and auxiliary variate is positive (high), the ratio method of estimation is used for estimating the population mean. On the other hand, if the correlation is negative, the product method of estimation envisaged by Robson (1957) is used. Sisodia and Dwivedi (1981) and Pandey and Dubey (1988) used coefficient of variation along with the population mean of auxiliary variate. Use of coefficient of kurtosis of auxiliary variate has also been in practice for improving the efficiency of the estimators of finite population mean. Upadhyaya and Singh (1999) and Singh et al. (2004) utilized coefficient of kurtosis of auxiliary variate for estimating the finite population mean. Rao and

[1] E-mail: tailorraj@gmail.com

Mudholkar (1967) and Singh and Espejo (2003) motivated authors to propose a ratio-cum-product estimator of finite population mean.

Let $U = (U_1, U_2 ..., U_N)$ be the finite population of size N and y and x be the study and auxiliary variates respectively. A sample of size n is drawn using simple random sampling without replacement to estimate the population mean $\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$ of study variate y.

The classical ratio and product estimators for $\overline{Y}$ are respectively given by

$$\overline{y}_R = \overline{y}\left(\frac{\overline{X}}{\overline{x}}\right) \tag{1.1}$$

and

$$\overline{y}_P = \overline{y}\left(\frac{\overline{x}}{\overline{X}}\right) \tag{1.2}$$

where $\overline{x} = \sum_{i=1}^{n} x_i / n$ and $\overline{y} = \sum_{i=1}^{n} y_i / n$. Here it is assumed that $\overline{X} = \sum_{i=1}^{N} x_i / N$, population mean of auxiliary variate is known.

Sisodia and Dwivedi (1981) suggested a ratio estimator of population men using coefficient of variation C_x of auxiliary variate as

$$\hat{\overline{Y}}_{SD} = \overline{y}\left(\frac{\overline{X} + C_x}{\overline{x} + C_x}\right) \tag{1.3}$$

Utilizing the information on coefficient of variation C_x and coefficient of kurtosis $\beta_2(x)$ of the auxiliary variate x, Upadhyaya and Singh (1999) proposed the following ratio and product estimators respectively

$$\hat{\overline{Y}}_{UP1} = \overline{y}\left(\frac{\overline{X}C_x + \beta_2(x)}{\overline{x}C_x + \beta_2(x)}\right) \tag{1.4}$$

$$\hat{\overline{Y}}_{UP2} = \overline{y}\left(\frac{\overline{x}C_x + \beta_2(x)}{\overline{X}C_x + \beta_2(x)}\right) \tag{1.5}$$

Singh et al. (2004) defined a ratio estimator using coefficient of kurtosis $\beta_2(x)$ which is

$$\hat{\overline{Y}}_S = \overline{y}\left(\frac{\overline{X} + \beta_2(x)}{\overline{x} + \beta_2(x)}\right) \tag{1.6}$$

2. Proposed Ratio-Cum-Product Estimator

The proposed ratio-cum-product estimator of population mean $\overline{Y}$ is

$$\hat{\overline{Y}}_M = \overline{y}\left[\alpha\left(\frac{\overline{X}C_x+\beta_2(x)}{\overline{x}C_x+\beta_2(x)}\right)+(1-\alpha)\left(\frac{\overline{x}C_x+\beta_2(x)}{\overline{X}C_x+\beta_2(x)}\right)\right] \tag{2.1}$$

where α is a suitably chosen scalar. It is to be noted that for α =1 and α = 0, $\hat{\overline{Y}}_M$ reduces to the estimators $\hat{\overline{Y}}_{UP1}$ and $\hat{\overline{Y}}_{UP2}$ respectively suggested by Upadhyaya and Singh (1999). Thus $\hat{\overline{Y}}_{UP1}$ and $\hat{\overline{Y}}_{UP2}$ are particular case of proposed estimator $\hat{\overline{Y}}_M$.

To obtain the bias and MSE of $\hat{\overline{Y}}_M$, we write
$\overline{y}=\overline{Y}(1+e_0)$ and $\overline{x}=\overline{X}(1+e_1)$ such that $E(e_0)=E(e_1)=0$ and

$$E(e_0^2)=\left(\frac{1}{n}-\frac{1}{N}\right)C_y^2,\qquad E(e_1^2)=\left(\frac{1}{n}-\frac{1}{N}\right)C_x^2,\qquad E(e_0e_1)=\left(\frac{1}{n}-\frac{1}{N}\right)\rho\, C_y\, C_x$$

where

$$C_y=S_y/\overline{Y},\qquad C_x=S_x/\overline{X},\qquad \rho=S_{yx}/S_xS_y,\qquad K=\rho C_y/C_x,$$

$$S_x^2=\sum_{i=1}^{N}(x_i-\overline{X})^2/(N-1),\qquad S_y^2=\sum_{i=1}^{N}(y_i-\overline{Y})^2/(N-1),$$
$$S_{xy}=\sum_{i=1}^{N}(y_i-\overline{Y})(x_i-\overline{X})/(N-1).$$

Expressing (2.1) in terms of $e_i^{'}s$ we have (i=1,2)

$$\hat{\overline{Y}}_M=\overline{Y}(1+e_0)\left[\alpha(1+t_3\,e_1)^{-1}+(1-\alpha)(1+t_3\,e_1)\right], \tag{2.2}$$

where $t_3=\overline{X}C_x\big/(\overline{X}C_x+\beta_2(x))$.

To the first degree of approximation, the bias and mean squared error of $\hat{\overline{Y}}_M$ are respectively given as

$$B(\hat{\overline{Y}}_M)=\frac{(1-f)}{n}\overline{Y}t_3\,C_x^2\left[K+t_3(t_3-2K)\right], \tag{2.3}$$

and

$$MSE(\hat{\bar{Y}}_M) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + (1-2\alpha)t_3\, C_x^2\{(1-2\alpha)t_3 + 2K\}\right]. \qquad (2.4)$$

Thus the estimator $\hat{\bar{Y}}_M$ with $\alpha = K/(2K - t_3)$ is almost unbiased. It is also observed from (2.3) that the bias of $\hat{\bar{Y}}_M$ is negligible for large sample.

Mean squared error of $\hat{\bar{Y}}_M$ in (2.4) is minimized for

$$\alpha = \frac{(t_3 + K)}{2t_3} = \alpha_0\,(\text{say}) \qquad (2.5).$$

Substitution of (2.5) in (2.1) yields the asymptotically optimum estimator (AOE) for

$$\bar{Y} \text{ as } \hat{\bar{Y}}_M^{(opt)} = \frac{\bar{y}}{2t_3}\left[(t_3 + K)\left(\frac{\bar{X}C_x + \beta_2(x)}{\bar{x}C_x + \beta_2(x)}\right) + (t_3 - K)\left(\frac{\bar{x}C_x + \beta_2(x)}{\bar{X}C_x + \beta_2(x)}\right)\right] \qquad (2.6)$$

Putting (2.5) in (2.3) and (2.4), we get the bias and variance of $\hat{\bar{Y}}_M^{(opt)}$ respectively as

$$B\left(\hat{\bar{Y}}_M^{(opt)}\right) = \frac{(1-f)}{2n}\bar{Y}\,C_x^2(t_3 - K)(t_3 + 2K), \qquad (2.7)$$

and

$$MSE\left(\hat{\bar{Y}}_M^{(opt)}\right) = \frac{(1-f)}{n}S_y^2(1-\rho^2). \qquad (2.8)$$

It is clear that mean squared error of $\hat{\bar{Y}}_M^{(opt)}$ is same as that of the approximate variance of the usual linear regression estimator $\bar{y}_{lr} = \bar{y} + \hat{\beta}(\bar{X} - \bar{x})$, where $\hat{\beta}$ is the sample regression coefficient of y on x.

3. Efficiency Comparisons

Under simple random sampling without replacement (SRSWOR), variance of sample mean $\bar{y}$ is

$$V(\bar{y}) = \frac{(1-f)}{n}\bar{Y}^2 C_y^2, \qquad (3.1)$$

and the variance of $\bar{y}_R$, $\bar{y}_P$, $\hat{\bar{Y}}_{SD}$, $\hat{\bar{Y}}_S$, $\hat{\bar{Y}}_{UP1}$ and $\hat{\bar{Y}}_{UP2}$ to the first degree of approximation are respectively given by

$$MSE(\bar{y}_R) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + C_x^2(1-2K)\right] \tag{3.2}$$

$$MSE(\bar{y}_P) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + C_x^2(1+2K)\right] \tag{3.3}$$

$$MSE(\hat{\bar{Y}}_{SD}) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + t_1\, C_x^2(t_1 - 2K)\right] \tag{3.4}$$

$$MSE(\hat{\bar{Y}}_S) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + t_2\, C_x^2(t_2 - 2K)\right] \tag{3.5}$$

$$MSE(\hat{\bar{Y}}_{UP2}) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + t_3\, C_x^2(t_3 - 2K)\right] \tag{3.6}$$

$$MSE(\hat{\bar{Y}}_{UP2}) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + t_3\, C_x^2(t_3 + 2K)\right] \tag{3.7}$$

From (2.4) and (3.1), it is observed that $\hat{\bar{Y}}_M$ is more efficient than usual unbiased estimator $\bar{y}$ if

$$\left.\begin{array}{ll} either & \frac{1}{2} < \alpha < \left(\frac{1}{2} + \frac{K}{t_3}\right) \\ or & \left(\frac{1}{2} + \frac{K}{t_3}\right) < \alpha < \frac{1}{2} \end{array}\right\} \tag{3.8}$$

Comparison of (2.4) and (3.2) shows that $\hat{\bar{Y}}_M$ is more efficient than usual ratio estimator

$$\bar{y}_R \quad \left.\begin{array}{ll} either & \frac{(1+t_3)}{2t_3} < \alpha < \left(\frac{t_3 + 2K - 1}{2t_3}\right) \\ or & \left(\frac{t_3 + 2K - 1}{2t_3}\right) < \alpha < \frac{(1+t_3)}{2t_3} \end{array}\right\} \tag{3.8}$$

From (2.4) and (3.3) it is clear that $\hat{\bar{\bar{Y}}}_M$ would be more efficient than $\bar{y}_P$ if

$$\left.\begin{array}{ll} either & \left(\dfrac{t_3+2K+1}{2t_3}\right)<\alpha<\dfrac{(t_3-1)}{2t_3} \\ or & \dfrac{(t_3-1)}{2t_3}<\alpha< \end{array}\right\} \tag{3.9}$$

Comparing (2.4) and (3.4), it is observed that $\hat{\bar{\bar{Y}}}_M$ is more efficient than Sisodia and Dwivedi (1981) estimator $\hat{\bar{\bar{Y}}}_{SD}$ if

$$\left.\begin{array}{ll} either & 1<\alpha<\dfrac{K}{t_1} \\ or & \dfrac{K}{t_1}<\alpha<1 \end{array}\right\} \tag{3.10}$$

Comparing (2.4) and (3.5), it is observed that $\hat{\bar{\bar{Y}}}_M$ is more efficient than Singh et al. (2004) $\hat{\bar{\bar{Y}}}_S$ if

$$\left.\begin{array}{ll} either & 1<\alpha<\dfrac{K}{t_2} \\ or & \dfrac{K}{t_2}<\alpha<1 \end{array}\right\} \tag{3.11}$$

Comparing (2.4) and (3.6), it is observed that $\hat{\bar{\bar{Y}}}_M$ is more efficient than Upadhyaya and Singh (1999) estimator $\hat{\bar{\bar{Y}}}_{UP1}$

$$\left.\begin{array}{ll} either & 1<\alpha<\dfrac{K}{t_3} \\ or & \dfrac{K}{t_3}<\alpha<1 \end{array}\right\} \tag{3.12}$$

Further, from (2.4) and (3.5), it is seen that $\hat{\bar{\bar{Y}}}_M$ is more efficient than Upadhyaya and Singh (1999) estimator $\hat{\bar{\bar{Y}}}_{UP2}$ if

$$\left.\begin{array}{ll} \textit{either} & 0<\alpha<\left(1+\dfrac{K}{t_3}\right) \\ \textit{or} & \left(1+\dfrac{K}{t_3}\right)<\alpha<0 \end{array}\right\} \qquad (3.13)$$

4. Empirical Study

To analyze the performance of the proposed estimator in comparison to other estimators, four natural population data sets are being considered. The descriptions of the populations are given below.

Population-I [Source: Das (1988)]

The population consists of 278 village town/wards under Gajole police station of Malada district of West Bengal, India (in fact, only the villages of town/wards which are shown as inhabited and common in both census 1961 and census 1971 lists have been considered). The variates considered are

x: The number of agricultural labourers for 1961

y: The number of agricultural labourers for 1971

$\overline{Y}$=39.0680, $\overline{X}$=25.1110, C_y=1.4451,

C_x=1.6198, ρ=0.7213, $\beta_2(x)$=38.8898

Population-II [Source: Das (1988)]

It consists of 142 cities of India with population (number of persons) 100,000 and above; the character x and y being

x: Census population in the year 1961

y: Census population in the year 1971

$\overline{Y}$=4015.2183, $\overline{X}$=2900.3872, C_y=2.1118,

C_x=2.1971, ρ=.9948, $\beta_2(x)$=48.1567

Population-III [Source: Cochran (1977)]

The variates are defined as follows:

y: The number of persons per block

x: The number of rooms per block

$\overline{Y}$=101.1, $\overline{X}$=58.80, C_y=0.14450,

C_x=0.1281 , ρ=0.6500, $\beta_2(x)$=2.2387.

Population-IV [Source: Pandey and Dube (1988)]

The population constants are as follows:

N=20, n=8,

$\overline{Y} = 19.55$ $\overline{X} = 18.8$, $C_y^2 = 0.1262$,

$C_x^2 = 0.1555$, $\rho_{yx} = -0.9199$, $\beta_2(x) = 3.0613$

Table 4.1. Range of α in which $\hat{\overline{Y}}_M$ is better than $\overline{y}$, $\overline{y}_R$, $\overline{y}_p$ $\hat{\overline{Y}}_{SD}$, $\hat{\overline{Y}}_S$, $\hat{\overline{Y}}_{UP1}$ and $\hat{\overline{Y}}_{UP2}$

Population	Range of α in which is $\hat{\overline{Y}}_M$ better than							Optimum value of α
	$\overline{y}$	$\overline{y}_R$	$\overline{y}_P$	$\hat{\overline{Y}}_{SD}$	$\hat{\overline{Y}}_S$	$\hat{\overline{Y}}_{UP1}$	$\hat{\overline{Y}}_{UP2}$	α_0
I	(0.5, 1.463)	(1.004, .989)	(-0.04, .1.967)	(1.00, .957)	(1.00, .972)	(.963, 1.00)	(0.0,1.963)	(0.982)
II	(0.5, 1.759)	(0.989, 0.787)	(-0.478,2.737)	(1.00, .685)	(1.00,1.64)	(1, 1.258)	(0.0, 2.59)	(1.129)
III	(0.5, 1.451)	(0.683, 0.803)	(-0.149, 2.10)	(1.00, .735)	(1.00, .761)	(0.951,1.00)	(0.0,1.951)	(0.976)
IV	(0.5, -.670)	(0.683, -1.378)	(-.064.8, .035)	(1.00, -.846)	(1.00, -.9632)	(1.00, -1.17)	(0.0,-.1709)	(-.0857)

Table 4.2. Percent relative efficiencies of $\overline{y}$, $\overline{y}_R$, $\overline{y}_p$ $\hat{\overline{Y}}_{SD}$, $\hat{\overline{Y}}_S$, $\hat{\overline{Y}}_{UP1}$, $\hat{\overline{Y}}_{UP2}$ and $\hat{\overline{Y}}_M$ or $\hat{\overline{Y}}_M^{(opt)}$ with respect to $\overline{y}$

Estimators	Population I	Population II	Population III	Population IV
$\overline{y}$	100.00	100.00	100.00	100.00
$\overline{y}_R$	156.40	8031.10	157.87	23.39
$\overline{y}_p$	*	*	*	526.45
$\hat{\overline{Y}}_{SD}$	169.57	8077.28	158.10	23.91
$\hat{\overline{Y}}_S$	178.90	8935.82	161.52	27.27
$\hat{\overline{Y}}_{UP1}$	199.32	8473.89	172.83	32.65
$\hat{\overline{Y}}_{UP2}$	*	*	*	582.03
$\hat{\overline{Y}}_M$ ($\hat{\overline{Y}}_M^{(opt)}$)	208.45	9640.45	173.16	650.26

* *Data are not applicable.*

Table 4.2 shows that there is a significant gain in efficiency by using proposed estimator $\hat{\bar{Y}}_M$ over unbiased estimator $\bar{y}$, usual ratio estimator $\bar{y}_R$, product estimator $\bar{y}_p$, Sisodiya and Dwivedi (1981) estimator $\hat{\bar{Y}}_{SD}$, Singh et al. (2004) estimator $\hat{\bar{Y}}_S$ and Upadhyay and Singh (1999) estimators $\hat{\bar{Y}}_{UP1}$ and $\hat{\bar{Y}}_{UP2}$.

Table 4.1 provides the wide range of α in which suggested estimator $\hat{\bar{Y}}_M$ ($\hat{\bar{Y}}_M^{(opt)}$) is more efficient then all estimators considered in this paper. This shows that even if the scalar α deviates from its optimum value (α_{opt}), the suggested estimator $\hat{\bar{Y}}_M^{(opt)}$ will yield better estimates than $\bar{y}$, $\bar{y}_R$, $\bar{y}_p$, $\hat{\bar{Y}}_{SD}$, $\hat{\bar{Y}}_S$, $\hat{\bar{Y}}_{UP1}$ and $\hat{\bar{Y}}_{UP2}$. Therefore, suggested estimator $\hat{\bar{Y}}_M$ is recommended for use in practice.

REFERENCES

COCHRAN, W.G. (1977). Sampling Techniques. Third U. S. Edition. Wiley Eastern Limited. 325.

DAS, A.K. (1988). Contribution to the theory of sampling strategies based on auxiliary information. Ph.D. Thesis, BCKV, West Bengal, India.

PANDEY, B.N. and DUBEY, V. (1988). Modified product estimator using coefficient of variation of auxiliary variable. Assam Stat. Rev., 2, 64–66.

RAO, P.S. R.S. and MUDHOLKAR, G.S. (1967). Generalised multivariate estimators for the mean of a finite population. Jour. Ind. Stat. Assoc., 62, 1009–1012.

ROBSON, D.S. (1957). Application of multivariate polykays to the theory of unbiased ratio-type estimation. Jour. Amer. Stat. Assoc., 59, 1225–1226.

SINGH, H.P., and ESPEJO, M.R. (2003). On linear regression and ratio estimator using coefficient of variation of auxiliary variate. Statistician, 52,1, 59–67.

SINGH, H.P. TAILOR, R. (2005). Estimation of finite population mean using known correlation coefficient between auxiliary characters. Statistica, Anno LXV, n.4, pp. 407–418.

SINGH, H.P., KAKRAN, TAILOR, R. and TAILOR, R. (2004). A modified ratio estimator of population mean. Jour. Ind. Soc. Agri. Stat., 33(1), 13–18.

SISODIA, B.V.S. and DWIVEDI, V.K. (1981). A modified ratio estimator using coefficient of variation of auxiliary variable. Jour. Ind. Soc. Agri. Stat., 33(1), 13–18.

UPADHYAYA, L.N. and SINGH, H.P. (1999). Use of transformed auxiliary variable in estimating the finite population mean. Biometrical Jour., 41(5), 627–636.

STATISTICS IN TRANSITION-new series, July 2009
Vol. 10, No. 1, pp. 25—58

SOME SHRINKAGE ESTIMATORS FOR ESTIMATING THE STANDARD DEVIATION AND ITS INVERSE FOR NORMAL PARENT

Housila P. Singh[1] and Vankim Chander[2]

ABSTRACT

Shrunken estimator has its importance in the small sample theory when appropriate prior information about the unknown parameter is supposed to be known. The present paper investigates classes of shrunken estimators for estimating standard deviation (σ) and its inverse (σ^{-1}) in the case of univariate normal parent. The need to study these parameters arises due to their importance to estimate different parametric functions such as Mean Deviation about Mean ($\sigma\sqrt{\left(2/\pi\right)}$), Process Capability Index (C_p), Standard Deviation (σ), Standard Error of Mean $\left(\frac{\sigma}{\sqrt{n}}\right)$, Coefficient of Variation (C.V) with known mean etc., using the prior information or guessed value of σ. Simulation studies confirm the high efficiency of the developed classes of shrunken estimators when compared with their usual unbiased estimators and minimum mean squared error (MMSE) estimators.

Key words: Bias, Gauss-Laguerre integration method, Mean Squared Error, Normal parent, Percent Relative Efficiency (PRE), Prior information.

1. Introduction

The normal distribution has dominated statistical practice as well as theory. Any variable whose expression results from the additive contribution of many small effects will tend to be normally distributed. For measurements whose distributions are not normal, a simple transformation of the scale of measurement may induce approximate normality. The square root, and logarithm ln(x) are often

[1] hpsujn@rediffmail.com : School of Studies in Vikram University, Ujjain, MP, India 456010.
[2] vcupadhyay06@yahoo.co.in : SAHARA Arts and Management Academy, Lucknow, UP, India.

used as such transformations, so normal distribution is very important in statistical point of views.

1.1.The Model

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n, drawn from population N(μ , σ^2), the probability distribution function (pdf) of which is given by

$$f(x;\mu,\sigma^2)=\begin{cases}\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} & ,-\infty<x<\infty\,,-\infty<\mu<\infty\,,\sigma>0\\ 0 & ,\quad otherwise\,.\end{cases} \tag{1.1}$$

μ is the mean and σ is the standard deviation.

1.2.Classical Estimators

It is desired to estimate standard deviation (S.D) σ and its inverse σ^{-1}.
The conventional unbiased estimators of σ and σ^{-1} are respectively given by

$$T_{(u,\sigma)} = c_1(n)s \tag{1.2}$$

and

$$T_{(u,\sigma^{-1})} = c_2(n)s^{-1} \;, \tag{1.3}$$

where

$$c_1(n) = \left(\frac{n-1}{2}\right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{(n-1)}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \;, \tag{1.4}$$

$$c_2(n) = \left(\frac{n-1}{2}\right)^{-\frac{1}{2}} \frac{\Gamma\left(\frac{(n-1)}{2}\right)}{\Gamma\left(\frac{(n-2)}{2}\right)} \tag{1.5}$$

and

$$s^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2 \quad \text{with} \quad \bar{x}=\frac{1}{n}\sum_{i=1}^{n}x \;.$$

The variance of $T_{(u,\sigma)}$ and $T_{(u,\sigma^{-1})}$ are respectively given by

$$Var\left\{T_{(u,\sigma)}\right\} = v_1(n)\sigma^2 \tag{1.6}$$

and

$$Var\left\{T_{\left(u,\sigma^{-1}\right)}\right\} = v_2(n)\left(1/\sigma^2\right) \tag{1.7}$$

where

$$v_1(n) = [(c_1(n))^2 - 1] \ , \tag{1.8}$$

$$v_2(n) = [c_3(n) - 1] \tag{1.9}$$

and

$$c_3(n) = \frac{\Gamma\left(\frac{n-3}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma^2\left(\frac{n-2}{2}\right)} \ . \tag{1.10}$$

The minimum mean squared error (MMSE) estimator in the class Q_b=bs (b, being suitably chosen constant such that mean squared error (MSE) of Q_b is minimum) is defined by

$$T_{(MMSE,\sigma)} = \frac{s}{c_1(n)} \ , \tag{1.11}$$

with bias and MSE

$$Bias\left\{T_{(MMSE,\sigma)}\right\} = -\sigma\left\{\frac{v_1(n)}{v_1(n)+1}\right\} \tag{1.12}$$

and

$$MSE\left\{T_{(MMSE,\sigma)}\right\} = \sigma^2\left\{\frac{v_1(n)}{v_1(n)+1}\right\} \tag{1.13}$$

respectively.

From (1.6) and (1.13), we have

$$Var\left\{T_{(u,\sigma)}\right\} - MSE\left\{T_{(MMSE,\sigma)}\right\} = \left\{\frac{(v_1(n))^2}{v_1(n)+1}\right\}\sigma^2 > 0 \ , \tag{1.14}$$

which gives the inequality

$$Var\left\{T_{(u,\sigma)}\right\} \;>\; MSE\left\{T_{(MMSE,\sigma)}\right\} . \tag{1.15}$$

Thus the MMSE estimator $T_{(MMSE,\sigma)}$ is more efficient than unbiased estimator $T_{(u,\sigma)}$.

Further the MMSE estimator in the class of estimators $d_{M^*}= \frac{M^*}{s}$ (M^*, being suitably chosen constant such that MSE of d_{M^*} is least) is given by

$$T_{\left(MMSE,\sigma^{-1}\right)} \;=\; c_4(n)\left(\frac{1}{s}\right) , \tag{1.16}$$

where $c_4(n) = \sqrt{\left(\frac{2}{n-1}\right)}\ \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-3}{2}\right)}$.

The bias and MSE are respectively given by

$$Bias\left\{T_{\left(MMSE,\sigma^{-1}\right)}\right\} = -\frac{1}{\sigma}\left\{\frac{v_2(n)}{v_2(n)+1}\right\} \tag{1.17}$$

and

$$MSE\left\{T_{\left(MMSE,\sigma^{-1}\right)}\right\} = \frac{1}{\sigma^2}\left\{\frac{v_2(n)}{v_2(n)+1}\right\} . \tag{1.18}$$

From (1.10) and (1.18) we have

$$Var\left\{T_{\left(u,\sigma^{-1}\right)}\right\} - MSE\left\{T_{\left(MMSE,\sigma^{-1}\right)}\right\} = \frac{1}{\sigma^2}\left\{\frac{\left\{v_2(n)\right\}^2}{v_2(n)+1}\right\} > 0 , \tag{1.19}$$

which yields the inequality

$$MSE\left\{T_{\left(MMSE,\sigma^{-1}\right)}\right\} < Var\left\{T_{\left(u,\sigma^{-1}\right)}\right\} . \tag{1.20}$$

Thus, the MMSE estimator $T_{\left(MMSE,\sigma^{-1}\right)}$ is more efficient than unbiased estimator $T_{\left(u,\sigma^{-1}\right)}$. Experimenter often possesses some knowledge of experimental conditions based on the acquaintance with the behaviour of the system under consideration or from past experience or from some extraneous source, and is thus in a position to give an adequate guess or an initial estimate, say θ_0 of the value of the unknown parameter θ. Thompson (1968 a, b) developed shrinkage estimator for the mean, Pandey and Singh (1977 a) studied the performance of the shrinkage estimator of the variance and many more authors Pandey (1979), Jani

(1991), Gangele and Singh (1994), Kourouklis (1994), Singh and Singh (1997), Singh et al. (1999, 2004), Singh and Saxena (2003 a, b, c, d) , Joshi (2005) , Singh and Chander (2007 a, b, c, d, e) have introduced shrinkage estimators for the parameters of normal/non-normal parents using prior knowledge of the unknown parameters.

In the present paper we have investigated some classes of shrinkage estimators for estimating standard deviation (σ) and its inverse (σ^{-1}) in the case of univariate normal parent. The problem of estimating these parameters arises due to their importance in estimating different parametric functions such as Mean Deviation about Mean $\left\{\sigma\sqrt{\left(\frac{2}{\pi}\right)}\right\}$, Process Capability Index (C_p), Standard Deviation (σ), Standard Error of Mean $\left\{\frac{\sigma}{\sqrt{n}}\right\}$, Coefficient of Variation (C.V) with known mean etc., using the prior information or guessed value of σ.

Simulation studies confirm the high efficiency of the developed classes of shrunken estimators when compared with their usual unbiased estimators and (MMSE) estimators.

2. The proposed class of shrinkage estimators for S.D. σ

Let σ_0 be the guessed value of σ available from the past experience. Then we define a class of estimators for σ as

$$T_{(s,\sigma)} = \ddot{k}\,T_{(u,\sigma)} + (1-\ddot{k})\,\sigma_0$$

$$= \ddot{k}\,(T_{(u,\sigma)} - \sigma_0) + \sigma_0 \quad , \tag{2.1}$$

where $\ddot{k}$ is a scalar such that $0 \le \ddot{k} \le 1$.

The bias and MSE of $T_{(s,\sigma)}$ are respectively given by

$$Bias\left\{T_{(s,\sigma)}\right\} = \left\{E\left(T_{(s,\sigma)}\right) - \sigma\right\} = \sigma\,(1-\ddot{k})(\hat{\lambda} - 1) \tag{2.2}$$

and

$$MSE\left\{T_{(s,\sigma)}\right\} = Var\left\{T_{(s,\sigma)}\right\} + \left\{Bias\left\{T_{(s,\sigma)}\right\}\right\}^2$$

$$= \ddot{k}^2\sigma^2[(c_1(n))^2 - 1] + \sigma^2(1-\ddot{k})^2(\hat{\lambda} - 1)^2$$

$$= \sigma^2\,[\ddot{k}^2[(c_1(n))^2 - 1] + (1-\ddot{k})^2(\hat{\lambda} - 1)^2]$$

$$= \sigma^2 [\ddot{k}^2 v_1(n) + (1-\ddot{k})^2 (\hat{\lambda}-1)^2] \ , \tag{2.3}$$

where $\hat{\lambda} = \dfrac{\sigma_0}{\sigma}$.

From (1.6) and (2.3), we have

$$Var\left\{T_{(u,\sigma)}\right\} - MSE\left\{T_{(s,\sigma)}\right\} = \sigma^2 [v_1(n) - \ddot{k}^2 v_1(n) - (1-\ddot{k})^2(\hat{\lambda}-1)^2]$$

$$= \sigma^2 [v_1(n)(1-\ddot{k}^2) - (1-\ddot{k})^2(\hat{\lambda}-1)^2] \, ,$$

which is positive if

$$[v_1(n)(1-\ddot{k}^2) - (1-\ddot{k})^2(\hat{\lambda}-1)^2] > 0 \, ,$$

$$\text{i.e. if } [v_1(n)(1+\ddot{k}) - (1-\ddot{k})(\hat{\lambda}-1)^2] > 0 \, , \ (1-\ddot{k}) > 0$$

$$\text{i.e. if } \ddot{k}\ [v_1(n) + (\hat{\lambda}-1)^2] > [(\hat{\lambda}-1)^2 - v_1(n)]$$

$$\text{i.e. if } \ddot{k} > \left\{\frac{\left(\hat{\lambda}-1\right)^2 - v_1(n)}{\left(\hat{\lambda}-1\right)^2 + v_1(n)}\right\}$$

$$\text{i.e. } \left\{\frac{\left(\hat{\lambda}-1\right)^2 - v_1(n)}{\left(\hat{\lambda}-1\right)^2 + v_1(n)}\right\} < \ddot{k} \le 1 \ . \tag{2.4}$$

Further, from (1.13) and (2.3), we have

$$MSE\left\{T_{(MMSE,\sigma)}\right\} - MSE\left\{T_{(s,\sigma)}\right\} = \sigma^2 \left\{\frac{v_1(n)}{1+v_1(n)}\right\}$$

$$-\ddot{k}^2 v_1(n) - (1-\ddot{k})^2(\hat{\lambda}-1)^2 > 0$$

$$\text{if } \ddot{k}^2 \left\{(1-\hat{\lambda}^2) + v_1(n)\right\} - 2\ddot{k}(\hat{\lambda}-1)^2 + (\hat{\lambda}-1)^2 - \left\{\frac{v_1(n)}{1+v_1(n)}\right\} > 0$$

$$\text{i.e. } (a-b) < \ddot{k} < (a+b) \tag{2.5}$$

where

$$a = \frac{(\hat{\lambda}-1)^2}{v_1(n) + (\hat{\lambda}-1)^2} ,$$

$$b = \frac{v_1(n)\sqrt{\hat{\lambda}(2-\hat{\lambda})}}{\sqrt{\{1+v_1(n)\}}\left\{v_1(n) + (\hat{\lambda}-1)^2\right\}} , 0<\hat{\lambda} < 2 .$$

2.1.Numerical Illustrations

To see the performance of the proposed estimator $T_{(s,\sigma)}$ over the usual unbiased estimator $T_{(u,\sigma)}$ and MMSE estimator $T_{(MMSE,\sigma)}$, we have computed the percentage relative efficiencies (PRE's) of $T_{(s,\sigma)}$ with respect to $T_{(u,\sigma)}$ and $T_{(MMSE,\sigma)}$ using the formulae:

$$PRE\left\{T_{(s,\sigma)}, T_{(u,\sigma)}\right\} = \frac{MSE\left\{T_{(u,\sigma)}\right\}}{MSE\left\{T_{(s,\sigma)}\right\}} * 100 = \frac{Var\left\{T_{(u,\sigma)}\right\}}{MSE\left\{T_{(s,\sigma)}\right\}} * 100$$

$$= \frac{1}{\ddot{k}^2 + \{v_1(n)\}^{-1}\left(1-\ddot{k}\right)^2\left(\hat{\lambda}-1\right)^2} * 100 \qquad (2.6)$$

and

$$PRE\left\{T_{(s,\sigma)}, T_{(MMSE,\sigma)}\right\} = \frac{MSE\left\{T_{(MMSE,\sigma)}\right\}}{MSE\left\{T_{(s,\sigma)}\right\}} * 100$$

$$= \frac{v_1(n)}{\left\{v_1(n)\ddot{k}^2 + \left(1-\ddot{k}\right)^2\left(\hat{\lambda}-1\right)^2\right\}\{1+v_1(n)\}} * 100 , \qquad (2.7)$$

for different values of $\ddot{k}$, $\hat{\lambda}$, n and findings are displayed in Table 2.2. It is observed from the Table 2.2 that:

(a) the proposed estimator $T_{(s,\sigma)}$ is better than the usual unbiased estimator $T_{(u,\sigma)}$ and the MMSE estimator $T_{(MMSE,\sigma)}$ with substantial gain in efficiency when $\hat{\lambda} \in [0.70,1.30]$ and $\ddot{k} \in [0.25,0.75]$.

(b) larger gain in efficiency is observed when the sample size n is small.

(c) for fixed $\ddot{k}$ and n, the value of PRE decreases as $\widehat{\lambda}$ goes away form unity.

(d) for fixed $\ddot{k}$ and $\widehat{\lambda}$, the value of PRE decreases as n increases.

(e) the gain in efficiency by using $T_{(s,\sigma)}$ over MMSE estimator $T_{(MMSE,\sigma)}$ is fewer than by using $T_{(s,\sigma)}$ over the unbiased estimator $T_{(u,\sigma)}$.

Thus, the proposed estimator is more beneficial when σ_0, the guessed value is in the vicinity of the true value of σ and when the sample size n is small. In practice, when the observations are expensive, such small sample sizes may be all that are available.

We have also computed the range of $\ddot{k}$ (for different values of n and $\widehat{\lambda}$) in which the suggested estimator $T_{(s,\sigma)}$ is better than $T_{(u,\sigma)}$ and $T_{(MMSE,\sigma)}$ using the expressions (2.4) and (2.5) respectively and the results are tabled in Table 2.1. It is observed from Table 2.1 that the effective range of $\ddot{k}$ become shorter as sample size n goes up for fixed $\widehat{\lambda}$, also for fixed n the effective range becomes shorter as $\widehat{\lambda}$ goes away from unity.

Table 2.1 Range of $\ddot{k}$ for which the proposed estimator $T_{(s,\sigma)}$ is better than $T_{(u,\sigma)}$ and MMSE estimator $T_{(MMSE,\sigma)}$

$\widehat{\lambda}\downarrow$ n→	3	5	7	9	11
0.3	(0.28, 1.0) **(0.42, 0.87)**	(0.58, 1.0) **(0.65, 0.93)**	(0.70, 1.00) **(0.75, 0.95)**	(0.77, 1.00) **(0.80, 0.96)**	(0.81, 1.00) **(0.84, 0.97)**
0.5	(0.00,1.00) **(0.08, 0.88)**	(0.31, 1.00) **(0.37, 0.94)**	(0.49, 1.00) **(0.53, 0.96)**	(0.59, 1.00) **(0.62, 0.97)**	(0.66, 1.00) **(0.69, 0.97)**
0.7	(0.00, 1.00) **(0.00, 0.88)**	(0.00, 1.00) **(0.00, 0.94)**	(0.02, 1.00) **(0.06, 0.96)**	(0.17, 1.00) **(0.20, 0.97)**	(0.27, 1.00) **(0.30, 0.97)**
0.8	(0.00,1.00) **(0.00, 0.89)**	(0.00, 1.00) **(0.00, 0.94)**	(0.00, 1.00) **(0.00, 0.96)**	(0.00, 1.00) **(0.00, 0.97)**	(0.00, 1.00) **(0.00, 0.98)**
0.9	(0.00, 1.00) **(0.00, 0.89)**	(0.00, 1.00) **(0.00, 0.94)**	(0.00, 1.00) **(0.00, 0.96)**	(0.00, 1.00) **(0.00, 1.00)**	(0.00, 1.00) **(0.00, 0.98)**
1	(0.00, 1.00) **(0.00, 0.89)**	(0.00, 1.00) **(0.00, 0.94)**	(0.00, 1.00) **(0.00, 0.96)**	(0.00, 1.00) **(0.00, 1.00)**	(0.00, 1.00) **(0.00, 0.98)**
1.1	(0.00, 1.00) **(0.00, 0.89)**	(0.00, 1.00) **(0.00, 0.94)**	(0.00, 1.00) **(0.00, 0.96)**	(0.00, 1.00) **(0.00, 1.00)**	(0.00, 1.00) **(0.00, 0.98)**
1.2	(0.00, 1.00) **(0.00, 0.89)**	(0.00, 1.00) **(0.00, 0.94)**	(0.00, 1.00) **(0.00, 0.96)**	(0.00, 1.00) **(0.00, 1.00)**	(0.00, 1.00) **(0.00, 0.98)**
1.3	(0.00, 1.00) **(0.00, 0.88)**	(0.00, 1.00) **(0.00, 0.94)**	(0.02, 1.00) **(0.06, 0.96)**	(0.17, 1.00) **(0.20, 0.97)**	(0.27, 1.00) **(0.30, 0.97)**
1.5	(0.00, 1.00) **(0.08, 0.88)**	(0.31, 1.00) **(0.37, 0.94)**	(0.49, 1.00) **(0.53, 0.96)**	(0.59, 1.00) **(0.62, 0.97)**	(0.66, 1.00) **(0.69, 0.97)**
1.7	(0.28, 1.00) **(0.42, 0.87)**	(0.58, 1.00) **(0.65, 0.93)**	(0.70, 1.00) **(0.75, 0.95)**	(0.77, 1.00) **(0.80, 0.96)**	(0.81, 1.00) **(0.84, 0.97)**
1.9	(0.50, 1.00) **(0.65, 0.85)**	(0.72, 1.00) **(0.80, 0.92)**	(0.81, 1.00) **(0.86, 0.94)**	(0.85, 1.00) **(0.90, 0.96)**	(0.88, 1.00) **(0.92, 0.97)**
2	(0.57, 1.00) **n.e.**	(0.77, 1.00) **n.e.**	(0.84, 1.00) **n.e.**	(0.88, 1.00) **n.e.**	(0.90, 1.00) **n.e.**

Note : a) The ***bold*** *figures in parentheses show the range of* $\ddot{k}$ *for which the proposed estimator* $T_{(s,\sigma)}$ *is better than MMSE estimator* $T_{(MMSE,\sigma)}$. *b)* ***n.e*** *stands for does not exit.*

Table 2.2 PRE of $T_{(s,\sigma)}$ with respect to unbiased estimator $T_{(u,\sigma)}$ and MMSE estimator $T_{(MMSE,\sigma)}$, for different values of n = 2,3,5,7,9 ; $\ddot{k}$ = 0.25 (0.25) 0.75 and $\widehat{\lambda}$

$\ddot{k}$	$\widehat{\lambda}$ ↓ n→	PRE of $T_{(s,\sigma)}$ w. r. t $T_{(u,\sigma)}$. 2	3	5	7	9	PRE of $T_{(s,\sigma)}$ w. r. t. $T_{(MMSE,\sigma)}$ 2	3	5	7	9
	0.3	183.36	93.35	46.42	30.78	23	116.73	73.32	41.02	28.33	21.61
	0.5	323.76	173.26	88.52	59.23	44.47	206.11	136.08	78.21	54.52	41.78
	0.7	661.41	403.59	223.87	154.37	117.71	421.07	316.98	197.8	142.08	110.6
	0.8	981.18	690.39	428.72	309.97	242/55	624.64	542.23	378.8	285.28	227.89
	0.9	1382.08	1203.57	950.67	784.12	666.9	879.86	945.28	839.99	721.7	626.59
0.25	1	1600	1600	1600	1600	1600	1018.59	1256.64	1413.72	1472.62	1503.3
	1.1	1382.08	1203.57	950.67	784.12	666.9	879.86	945.28	839.99	721.7	626.59
	1.2	981.18	690.39	428.72	309.97	242/55	624.64	542.23	378.8	285.28	227.89
	1.3	661.41	403.59	223.87	154.37	117.71	421.07	316.98	197.8	142.08	110.6
	1.5	323.76	173.26	88.52	59.23	44.47	206.11	136.08	78.21	54.52	41.78
	1.7	183.36	93.35	46.42	30.78	23	116.73	73.32	41.02	28.33	21.61
	Range of λ	[0.06,1.94]	[0.33,1.67]	[0.54,1.46]	[0.63,1.37]	[0.68,1.32]	[0.24,1.76]	[0.41,1.59]	[0.57,1.43]	[0.64,1.36]	[0.69,1.31]
	0.3	215.23	143.2	84.77	60.02	46.42	137.02	112.47	74.9	55.24	43.61
	0.5	278.17	208.89	138.06	102.82	81.86	177.09	164.06	121.99	94.64	76.91
	0.7	345.52	300.89	237.67	196.03	166.72	219.97	236.32	210	180.42	156.65
	0.8	373.8	348.92	306.85	273.52	246.63	237.97	274.04	271.13	251.74	231.73
	0.9	393.11	385.88	371.78	358.55	346.18	250.26	303.07	328.5	330	325.26
0.5	1	400	400	400	400	400	254.65	314.16	353.43	368.16	375.83
	1.1	393.11	385.88	371.78	358.55	346.18	250.26	303.07	328.5	330	325.26
	1.2	373.8	348.92	306.85	273.52	246.63	237.97	274.04	271.13	251.74	231.73
	1.3	345.52	300.89	237.67	196.03	166.72	219.97	236.32	210	180.42	156.65
	1.5	278.17	208.89	138.06	102.82	81.86	177.09	164.06	121.99	94.64	76.91
	1.7	215.23	143.2	84.77	60.02	46.42	137.02	112.47	74.9	55.24	43.61
	Range of λ	[0,2.3]	[0.1,0.9]	[0.38,1.62]	[0.5,1.5]	[0.57,1.43]	[0.07,1.93]	[0.24,1.76]	[0.43,1.57]	[0.52,1.48]	[0.58,1.42]
	0.3	162.3	148.24	125.8	109.1	96.28	103.32	116.43	111.15	100.42	90.46
	0.5	169.53	161.37	146.83	134.56	124.16	107.92	126.74	129.73	123.85	116.66
	0.7	174.72	171.5	165.24	159.35	153.86	111.23	134.7	146	146.67	144.56
	0.8	176.4	174.93	172	169.1	166.29	112.3	137.39	151.95	155.63	156.24
	0.9	177.43	177.06	176.29	175.52	174.76	112.96	139.06	155.77	161.55	164.2
0.75	1	177.78	177.78	177.78	177.78	177.78	113.18	139.63	157.08	163.62	167.03
	1.1	177.43	177.06	176.29	175.52	174.76	112.96	139.06	155.77	161.55	164.2
	1.2	176.4	174.93	172	169.1	166.29	112.3	137.39	151.95	155.63	156.24
	1.3	174.72	171.5	165.24	159.35	153.86	111.23	134.7	146	146.67	144.56
	1.5	169.53	161.37	146.83	134.56	124.16	107.92	126.74	129.73	123.85	116.66
	1.7	162.3	148.24	125.8	109.1	96.28	103.32	116.43	111.15	100.42	90.46
	Range of λ	[0,2.99]	[0,2.3]	[0.04,1.96]	[0.24,1.76]	[0.34,1.66]	[0.18,1.82]	[0.18,1.82]	[0.18,1.82]	[0.3,1.7]	[0.38,1.62]

3. Some modified shrinkage estimators for S.D. σ

The $MSE\left\{T_{(s,\sigma)}\right\}$ given by equation (2.3) is minimised for

$$k_{opt}^{(a)} = \frac{\left(1-\hat{\lambda}\right)^2}{\left(1-\hat{\lambda}\right)^2 + v_1(n)}$$

$$= \frac{(\sigma-\sigma_0)^2}{(\sigma-\sigma_0)^2 + \sigma^2 v_1(n)} \tag{3.1}$$

Now, replacing σ and σ^2 by their unbiased estimator $T_{(u,\sigma)}$ and $T_{\left(u,\sigma^2\right)}$ = s^2 respectively in (3.1), we get a consistent estimate of $k_{opt}^{(a)}$ as

$$\hat{k}_{opt}^{(1)} = \frac{\left(T_{(u,\sigma)}-\sigma_0\right)^2}{\left(T_{(u,\sigma)}-\sigma_0\right)^2 + v_1(n)s^2} \ . \tag{3.2}$$

Replacing σ by its unbiased estimator $T_{(u,\sigma)}$ in (3.1), we get

$$\hat{k}_{opt}^{(2)} = \frac{\left(T_{(u,\sigma)}-\sigma_0\right)^2}{\left(T_{(u,\sigma)}-\sigma_0\right)^2 + v_1(n)\left\{1+v_1(n)\right\}s^2} \ , \tag{3.3}$$

and replacing σ by its unbiased estimator $T_{(u,\sigma)}$ and σ^2 by its MMSE estimator $T_{\left(MMSE,\sigma^2\right)} = \left(\frac{n-1}{n+1}\right)s^2$ in (3.1), we get

$$\hat{k}_{opt}^{(3)} = \frac{\left(T_{(u,\sigma)}-\sigma_0\right)^2}{\left(T_{(u,\sigma)}-\sigma_0\right)^2 + v_1(n)\left(\frac{n-1}{n+1}\right)s^2} \ . \tag{3.4}$$

Many more consistent estimators of $k_{opt}^{(a)}$ can be obtained. However, a more flexible estimator of $k_{opt}^{(a)}$ can be obtained as :

$$\hat{k}_{opt}^{(\alpha)} = \frac{\left(T_{(u,\sigma)} - \sigma_0\right)^2}{\left(T_{(u,\sigma)} - \sigma_0\right)^2 + \alpha_1 s^2} \quad , \tag{3.5}$$

where $\alpha_1 (\geq 0)$ is a suitably chosen constant.

Thus, the resulting modified shrinkage estimators of S.D. σ are given by

$$T_{(s,\sigma)}^{(1)} = \sigma_0 + \frac{\left(T_{(u,\sigma)} - \sigma_0\right)^3}{\left(T_{(u,\sigma)} - \sigma_0\right)^2 + v_1(n) s^2} \tag{3.6}$$

$$T_{(s,\sigma)}^{(2)} = \sigma_0 + \frac{\left(T_{(u,\sigma)} - \sigma_0\right)^3}{\left(T_{(u,\sigma)} - \sigma_0\right)^2 + v_1(n)\left(1 + v_1(n)\right) s^2} \quad , \tag{3.7}$$

$$T_{(s,\sigma)}^{(3)} = \sigma_0 + \frac{\left(T_{(u,\sigma)} - \sigma_0\right)^3}{\left(T_{(u,\sigma)} - \sigma_0\right)^2 + \left(\frac{n-1}{n+1}\right) v_1(n) s^2} \quad , \tag{3.8}$$

and

$$T_{(s,\sigma)}^{(4)} = \sigma_0 + \frac{\left(T_{(u,\sigma)} - \sigma_0\right)^3}{\left(T_{(u,\sigma)} - \sigma_0\right)^2 + \alpha_1 s^2} \quad . \tag{3.9}$$

The biases and MSEs of the estimators $T_{(s,\sigma)}^{(i)}$, i = 1 to 4 are respectively given by

$$Bias\left\{T_{(s,\sigma)}^{(i)}\right\} = \left(\sigma_0 - \sigma\right) + \int_0^\infty \left[\frac{\left(T_{(u,\sigma)} - \sigma_0\right)^3}{\left\{\left(T_{(u,\sigma)} - \sigma_0\right)^2 + j s^2\right\}}\right] f(s) ds \tag{3.10}$$

and

$$MSE\left\{T_{(s,\sigma)}^{(i)}\right\} = \left(\sigma_0 - \sigma\right)^2 + 2\left(\sigma_0 - \sigma\right) \int_0^\infty \left[\frac{\left(T_{(u,\sigma)} - \sigma_0\right)^3}{\left\{\left(T_{(u,\sigma)} - \sigma_0\right)^2 + j s^2\right\}}\right] f(s) ds$$

$$+\int_0^\infty \left[\frac{\left(T_{(u,\sigma)}-\sigma_0\right)^6}{\left\{\left(T_{(u,\sigma)}-\sigma_0\right)^2+js^2\right\}^2} \right] f(s)\,ds \quad , \tag{3.11}$$

where $j = v_1(n),\ \ v_1(n)\{1 + v_1(n)\},\ \ v_1(n)\left(\frac{n-1}{n+1}\right)$ and $\alpha_1(\geq 0)$.

Now, transform the above expression in the form of $\widehat{\lambda}$, for this we are using transformation $y = \frac{(n-1)s^2}{2\sigma^2}$, we have

$$MSE\left\{T^{(i)}_{(s,\sigma)}\right\} =$$

$$\sigma^2 \left\{ \left(\widehat{\lambda}-1\right)^2 + \frac{2\left(\widehat{\lambda}-1\right)}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty \left[\frac{\left\{C_5(n)y^{\frac{1}{2}}-\widehat{\lambda}\right\}^3}{\left(C_5(n)y^{\frac{1}{2}}-\widehat{\lambda}\right)^2+\left(\frac{2}{n-1}\right)jy} \right] e^{-y} y^{\left(\frac{n-3}{2}\right)} dy \right.$$

$$\left. + \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty \left[\frac{\left\{C_5(n)y^{\frac{1}{2}}-\widehat{\lambda}\right\}^6}{\left\{\left(C_5(n)y^{\frac{1}{2}}-\widehat{\lambda}\right)^2+\left(\frac{2}{n-1}\right)jy\right\}^2} \right] e^{-y} y^{\left(\frac{n-3}{2}\right)} dy \right\} \tag{3.12}$$

where

$$C_5(n) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \ .$$

The PRE of $T^{(i)}_{(s,\sigma)}$ w.r.t. unbiased estimator $T_{(u,\sigma)}$ and MMSE estimator $T_{(MMSE,\sigma)}$ for different values of i = 1, 2, 3 and $j = v_1(n),\ \ v_1(n)\{1+v_1(n)\}$ and $v_1(n)\left(\frac{n-1}{n+1}\right)$ are given in the Tables 3.1, 3.2

and 3.3 respectively. It is observed that the proposed modified estimator $T^{(i)}_{(s,\sigma)}$ is better than $T_{(u,\sigma)}$ and $T_{(MMSE,\sigma)}$ when $\hat{\lambda} \in [0.7, 1.3]$ and $2 \le n \le 9$. For fixed $\hat{\lambda}$, the $PRE\{T^{(1)}_{(s,\sigma)}, T_{(u,\sigma)}\}$ and $PRE\{T^{(1)}_{(s,\sigma)}, T_{(MMSE,\sigma)}\}$ decrease as n increases and for fixed n these values decrease as $\hat{\lambda}$ goes away from unity where PRE's attain its maximum. Similar trend is found for $T^{(2)}_{(s,\sigma)}$ and $T^{(3)}_{(s,\sigma)}$ also. We also observe that the gain in efficiency w.r.t. $T_{(MMSE,\sigma)}$ is lesser than $T_{(u,\sigma)}$. Tables 3.1, 3.2 and 3.3 show that the estimator $T^{(2)}_{(s,\sigma)}$ is the best (in the sense of having smallest MSE) among $T^{(1)}_{(s,\sigma)}, T^{(2)}_{(s,\sigma)}$ and $T^{(3)}_{(s,\sigma)}$ followed by $T^{(3)}_{(s,\sigma)}$. Thus, the modified estimator $T^{(2)}_{(s,\sigma)}$ is to be preferred in practice when $\hat{\lambda}$ moves in the vicinity of unity and larger efficiency is observed when sample size n is small.

All the integration and gamma values are calculated using Gauss-Laguerre integration formula (10 point) with the help of computer programs developed in C++ platform.

Table 3.1 $PRE\{T^{(1)}_{(s,\sigma)}, T_{(u,\sigma)}\}$ and $PRE\{T^{(1)}_{(s,\sigma)}, T_{(MMSE,\sigma)}\}$

$\hat{\lambda}\downarrow$ n→	2	3	5	7	9
0.4	221.09	136.16	116.19	99.79	97.34
	140.75	**106.94**	**102.66**	**92.77**	**91.45**
0.5	251.83	166.21	117.54	99.96	98.49
	160.32	**130.54**	**103.86**	**92.01**	**92.53**
0.7	376.11	263.07	131.36	151.07	119.63
	239.44	**206.62**	**116.06**	**139.04**	**112.4**
0.8	489.72	275.15	163.78	180.54	136.59
	311.76	**216.11**	**144.72**	**166.17**	**128.34**
0.9	605.7	272.57	199.68	195.89	185.67
	385.6	**214.08**	**176.43**	**180.29**	**174.45**
1	650.39	263.12	211.21	204	211.98
	414.05	**206.66**	**186.62**	**187.76**	**199.17**
1.1	613.81	244.21	203.74	176.27	196.91
	390.76	**191.81**	**180.02**	**162.24**	**185.01**
1.2	542.48	213.59	180.2	141.54	156.23
	345.35	**167.76**	**159.22**	**130.27**	**146.79**
1.3	464.72	178.39	145.56	121.04	111.02
	295.85	**140.11**	**128.61**	**111.41**	**104.31**
1.5	321.31	127.85	94.51	89.13	77.01
	204.55	**100.41**	**83.51**	**82.04**	**72.36**
1.7	212.99	104.99	77.25	71.32	70.49
	135.6	**82.46**	**68.26**	**65.65**	**66.23**
Range of $\hat{\lambda}$	[0 , 2.1]	[0.2 , 1.7]	[0.25 , 1.42]	[0.57 , 1.42]	[0.55 , 1.32]
	[0.1 , 1.85]	**[0.3 , 1.5]**	**[0.37 , 1.36]**	**[0.6 , 1.36]**	**[0.6 , 1.31]**

Table 3.2 $PRE\{T^{(2)}_{(s,\sigma)}, T_{(u,\sigma)}\}$ and $PRE\{T^{(2)}_{(s,\sigma)}, T_{(MMSE,\sigma)}\}$

$\hat{\lambda}\downarrow$ n→	2	3	5	7	9
0.4	244.45	140.73	116	97.76	96.85
	155.62	**110.53**	**102.5**	**92.25**	**91**
0.5	292.32	175.48	118.1	99.78	98.21
	186.1	**137.83**	**104.35**	**91.84**	**92.27**
0.7	508.15	294.55	135.09	152.63	120.29
	323.5	**231.34**	**119.36**	**140.48**	**113.02**
0.8	729.82	318.93	170.12	185.01	138.98
	464.62	**250.49**	**150.31**	**170.28**	**130.58**
0.9	984.12	320.8	208.46	203.88	190.35
	626.51	**251.95**	**184.19**	**187.65**	**178.84**
1	1062.06	308.45	220.57	213.09	217.32
	676.13	**242.26**	**194.89**	**196.12**	**204.19**
1.1	913.93	280.45	211.49	182.15	200.31
	581.83	**220.26**	**186.86**	**167.65**	**188.21**
1.2	715.82	238.31	185.43	143.74	157.83
	455.7	**187.17**	**163.84**	**132.3**	**148.29**
1.3	554.45	193.51	148.68	121.03	111.71
	352.97	**151.98**	**131.37**	**111.4**	**104.96**
1.5	341.86	131.48	94.86	88.44	76.34
	217.63	**103.26**	**83.82**	**81.4**	**71.72**
1.7	217.6	103.1	75.86	70.39	69.5
	138.53	**80.97**	**67.02**	**64.79**	**65.3**
Range of $\hat{\lambda}$	[0 , 2]	[0.08 , 1.72]	[0.1 , 1.47]	[0.51 , 1.42]	[0.52 , 1.33]
	[0.06 , 1.85]	**[0.3 , 1.52]**	**[0.38 , 1.51]**	**[0.58 , 1.35]**	**[0.59 , 1.31]**

*Note: **Bold** figures denote the PREs/ranges w.r.t. MMSE estimator*

Table 3.3 $PRE\{T^{(3)}_{(s,\sigma)}, T_{(u,\sigma)}\}$ and $PRE\{T^{(3)}_{(s,\sigma)}, T_{(MMSE,\sigma)}\}$

$\hat{\lambda}\downarrow$ n→	2	3	5	7	9
0.4	164.72	122.18	114.92	102.04	98.65
	104.86	**95.96**	**101.54**	**93.91**	**92.69**
0.5	176.47	139.68	114.61	100.19	99.09
	112.35	**109.71**	**101.27**	**92.22**	**93.1**
0.7	209.87	191.84	120.46	145.15	117.24
	133.61	**150.67**	**106.44**	**133.59**	**110.16**
0.8	240.44	191.66	145.36	166.05	128.78
	153.07	**150.53**	**128.44**	**152.83**	**120.99**
0.9	271.08	188.67	174.48	172.53	170.61
	172.58	**148.18**	**154.17**	**158.79**	**160.3**
1	286.91	185.34	184.12	177.95	194.78
	182.65	**145.57**	**162.69**	**163.78**	**183.01**
1.1	291.1	179.22	180.25	158.82	185.33
	185.32	**140.76**	**159.27**	**146.17**	**174.13**
1.2	287.18	165.85	163.51	134.87	150.46
	182.83	**130.26**	**144.48**	**124.13**	**141.37**
1.3	275.62	146.84	135.35	121.15	108.45
	175.46	**115.33**	**119.59**	**111.5**	**101.9**
1.5	233.62	120.73	93.61	91.07	79.49
	148.73	**94.82**	**82.71**	**83.82**	**74.68**
1.7	179.12	111.31	82.05	74.46	73.91
	114.03	**87.43**	**72.5**	**68.53**	**69.44**
Range of $\hat{\lambda}$	[0.001 , 2]	[0.01 , 1.89]	[0.1 , 1.44]	[0.45 , 1.43]	[0.53 , 1.30]
	[0.35 , 1.77]	**[0.5 , 1.40]**	**[0.58 , 1.37]**	**[0.6 , 1.35]**	**[0.6 , 1.3]**

*Note: **Bold** figures denote the PREs/ranges w.r.t. MMSE estimator*

Expression (3.1) shows that the optimum value of $\ddot{k}$ depends on the unknown parameter σ. Since σ_0 is the guessed or prior value of σ, one may replace σ by $\dot{\alpha}\sigma_0$, where $\dot{\alpha}$ is a positive constant. Thus, putting $\sigma = \dot{\alpha}\sigma_0$ ($\dot{\alpha} > 0$), in (3.1), we get

$$k^{(b)}_{opt} = \frac{(\dot{\alpha}-1)^2}{(\dot{\alpha}-1)^2 + v_1(n)\dot{\alpha}^2} \quad .$$

Thus, the resulting estimator of σ is

$$T^{(*,opt)}_{(s,\sigma)} = \frac{(1-\dot{\alpha})^2 T_{(u,\sigma)} + v_1(n)\sigma_0\dot{\alpha}^2}{(1-\dot{\alpha})^2 + v_1(n)\dot{\alpha}^2} \quad , \tag{3.13}$$

The bias of $T_{(s,\sigma)}^{(*,opt)}$ is given by

$$Bias\left\{T_{(s,\sigma)}^{(*,opt)}\right\} = \mathrm{E}(T_{(s,\sigma)}^{(*,opt)}) - \sigma$$

$$= -\frac{\left(1-\hat{\lambda}\right)v_1(n)\sigma\,\dot{\alpha}^2}{\left(1-\dot{\alpha}\right)^2 + v_1(n)\dot{\alpha}^2} .$$

The MSE of $T_{(s,\sigma)}^{(*,opt)}$ is given by

$$MSE\left\{T_{(s,\sigma)}^{(*,opt)}\right\} = \mathrm{Var}\{T_{(s,\sigma)}^{(*,opt)}\} + \left[Bias\left\{T_{(s,\sigma)}^{(*,opt)}\right\}\right]^2$$

$$= \frac{\left(1-\dot{\alpha}\right)^4 v_1(n)\sigma^2}{\left\{\left(1-\dot{\alpha}\right)^2 + v_1(n)\dot{\alpha}^2\right\}^2} + \frac{\left(1-\hat{\lambda}\right)^2\left(v_1(n)\right)^2\dot{\alpha}^4\sigma^2}{\left\{\left(1-\dot{\alpha}\right)^2 + v_1(n)\dot{\alpha}^2\right\}^2}$$

$$= \frac{\sigma^2 v_1(n)\left\{\left(1-\dot{\alpha}\right)^4 + \left(1-\hat{\lambda}\right)^2 v_1(n)\dot{\alpha}^4\right\}}{\left\{\left(1-\dot{\alpha}\right)^2 + v_1(n)\dot{\alpha}^2\right\}^2} ,$$

which is smaller than the variance of the conventional unbiased estimator $T_{(u,\sigma)}$ if

$$\frac{\left\{\left(1-\dot{\alpha}\right)^4 + \left(1-\hat{\lambda}\right)^2 v_1(n)\dot{\alpha}^4\right\}}{\left\{\left(1-\dot{\alpha}\right)^2 + v_1(n)\dot{\alpha}^2\right\}^2} < 1$$

i.e. if $\dot{\alpha} \leq \left[1 + \sqrt{\frac{1}{2}\left\{\left(1-\hat{\lambda}\right)^2 - v_1(n)\right\}}\right]^{-1}$

which shows that $\dot{\alpha}$ should be between zero and one (i.e. $0 < \dot{\alpha} \leq 1$) for gain in efficiency.

4. The suggested class of shrinkage estimators for inverse of standard deviation σ^{-1}

Let us define a class of estimator for σ^{-1} as

$$T_{(s,\sigma^{-1})} = w\, T_{(u,\sigma^{-1})} + (1-w)\,\sigma_0^{-1}$$

$$= w\,(T_{(u,\sigma^{-1})} - \sigma_0^{-1}) + \sigma_0^{-1} \quad , \tag{4.1}$$

where w is a scalar such that $0 \le w \le 1$.

The bias and MSE of $T_{(s,\sigma^{-1})}$ are respectively given by

$$Bias\left\{T_{(s,\sigma^{-1})}\right\} = \mathrm{E}\left(T_{(s,\sigma^{-1})}\right) - \sigma^{-1}$$

$$= \sigma^{-1}(1-w)(\lambda^{*}-1) \tag{4.2}$$

and

$$MSE\left\{T_{(s,\sigma^{-1})}\right\} = Var\left\{T_{(s,\sigma^{-1})}\right\} + \left[Bias\left\{T_{(s,\sigma^{-1})}\right\}\right]^2$$

$$= \sigma^{-2}\,[\,w^2\, v_2(n) + (1-w)^2(\lambda^{*}-1)^2] \ , \tag{4.3}$$

where $\lambda^{*} = \lambda^{-1}$.

From (1.7) and (4.3), we have

$$Var\left\{T_{(s,\sigma^{-1})}\right\} - MSE\left\{T_{(s,\sigma^{-1})}\right\} = \sigma^{-2}\,[\,v_2(n)\,(1\text{-}\,w^2) - (1-w)^2(\lambda^{*}-1)^2] \, ,$$

which is positive if

$$[\,v_2(n)\,(1-w^2) - (1-w)^2(\lambda^{*}-1)^2] > 0 \, ,$$

$$\text{i.e. if } \ w > \left[\frac{\left\{(\lambda^{*}-1)^2 - v_2(n)\right\}}{\left\{(\lambda^{*}-1)^2 + v_2(n)\right\}}\right]$$

$$\text{i.e. } \ \left[\frac{\left\{(\lambda^{*}-1)^2 - v_2(n)\right\}}{\left\{(\lambda^{*}-1)^2 + v_2(n)\right\}}\right] < w \le 1 \ . \tag{4.4}$$

Further, from (1.18) and (4.3), we have

$$MSE\left\{T_{\left(MMSE,\sigma^{-1}\right)}\right\} - MSE\left\{T_{\left(s,\sigma^{-1}\right)}\right\} > 0 ,$$

$$\text{if } (c\text{–}d) < w < (c+d) , \tag{4.5}$$

where

$$c = \frac{(\lambda^{*}-1)^{2}}{\left\{v_{2}(n) + (\lambda^{*}-1)^{2}\right\}} ,$$

$$d = \frac{v_{2}(n)\sqrt{\lambda^{*}(2-\lambda^{*})}}{\sqrt{\{1+v_{2}(n)\}}\left\{v_{2}(n)+(\lambda^{*}-1)^{2}\right\}} , 0<\lambda^{*}<2 .$$

4.1.PRE and Empirical Study

To see the performance of the proposed estimator $T_{\left(s,\sigma^{-1}\right)}$ over the usual unbiased estimator $T_{\left(u,\sigma^{-1}\right)}$ and MMSE estimator $T_{\left(MMSE,\sigma^{-1}\right)}$, we have computed the percentage relative efficiencies (PRE's) of $T_{\left(s,\sigma^{-1}\right)}$ with respect to $T_{\left(u,\sigma^{-1}\right)}$ and $T_{\left(MMSE,\sigma^{-1}\right)}$ using the formulae:

$$PRE\left\{T_{\left(s,\sigma^{-1}\right)}, T_{\left(u,\sigma^{-1}\right)}\right\} = \frac{MSE\left\{T_{\left(u,\sigma^{-1}\right)}\right\}}{MSE\left\{T_{\left(s,\sigma^{-1}\right)}\right\}}*100 = \frac{Var\left\{T_{\left(u,\sigma^{-1}\right)}\right\}}{MSE\left\{T_{\left(s,\sigma^{-1}\right)}\right\}}*100$$

$$= \frac{1}{\left[w^{2} + \{v_{2}(n)\}^{-1}(1-w)^{2}\{\lambda^{*}-1\}^{2}\right]} \tag{4.6}$$

and

$$PRE\left\{T_{\left(s,\sigma^{-1}\right)}, T_{\left(MMSE,\sigma^{-1}\right)}\right\} = \frac{MSE\left\{T_{\left(MMSE,\sigma^{-1}\right)}\right\}}{MSE\left\{T_{\left(s,\sigma^{-1}\right)}\right\}}*100$$

$$= \frac{v_{2}(n)}{\left[v_{2}(n)w^{2} + (1-w)^{2}\{\lambda^{*}-1\}^{2}\right]\{1+v_{2}(n)\}}*100 , \tag{4.7}$$

for different w, λ^*, n and findings are given in Table 4.2. It is observed that:

- the proposed estimator $T_{(s,\sigma^{-1})}$ is better than the usual unbiased estimator $T_{(u,\sigma^{-1})}$ and MMSE estimator $T_{(MMSE,\sigma^{-1})}$ with substantial gain in efficiency when $\lambda^* \in [0.70,1.30] \Rightarrow \lambda \in [0.77,1.43]$ and $w \in [0.25,0.75]$.
- larger gain in efficiency is observed when sample size n is small.
- for fixed w and λ^*, the value of PRE decreases as n increases.
- for fixed w and n, the value of PRE decreases as λ^* goes away form unity.
- the gain in efficiency by using $T_{(s,\sigma^{-1})}$ over MMSE estimator $T_{(MMSE,\sigma^{-1})}$ is fewer than by using $T_{(s,\sigma^{-1})}$ over unbiased estimator $T_{(u,\sigma^{-1})}$.

Thus, the proposed estimator is more beneficial when σ_0, the guessed value is close to the true value of σ and when the sample size n is small.

The ranges of w (for different values of n and λ^*) in which $T_{(s,\sigma^{-1})}$ is better than $T_{(u,\sigma^{-1})}$ and $T_{(MMSE,\sigma^{-1})}$ are given in the Table 4.1 (using equations (4.4) and (4.5)). It is observed that the effective range of w becomes shorter as sample size n goes up for fixed λ^*, also for fixed n the effective range becomes shorter as λ^* goes away from unity.

Table 4.1 Range of w for which the proposed estimator $T_{\left(s,\sigma^{-1}\right)}$ is better than $T_{\left(u,\sigma^{-1}\right)}$ and MMSE estimator $T_{\left(MMSE,\sigma^{-1}\right)}$.

λ*↓ n→	7	9	11	13	15
0.4	(0.46,1.00) **(0.53, 0.93)**	(0.61,1.00) **(0.66, 0.95)**	(0.70,1.00) **(0.73, 0.97)**	(0.75,1.00) **(0.78, 0.97)**	(0.79,1.00) **(0.81, 0.98)**
0.5	(0.31,1.00) **(0.37, 0.94)**	(0.49,1.00) **(0.53, 0.96)**	(0.59,1.00) **(0.62, 0.97)**	(0.66,1.00) **(0.69, 0.97)**	(0.71,1.00) **(0.73, 0.98)**
0.7	(0.00,1.00) **(0.00, 0.94)**	(0.02,1.00) **(0.06, 0.96)**	(0.17,1.00) **(0.20, 0.97)**	(0.27,1.00) **(0.30, 0.97)**	(0.36,1.00) **(0.38, 0.98)**
0.8	(0.00,1.00) **(0.00, 0.94)**	(0.00,1.00) **(0.00, 0.96)**	(0.00,1.00) **(0.00, 0.97)**	(0.00,1.00) **(0.00, 0.98)**	(0.00,1.00) **(0.00, 0.98)**
0.9	(0.00,1.00) **(0.00, 0.94)**	(0.00,1.00) **(0.00, 0.96)**	(0.00,1.00) **(0.00, 1.00)**	(0.00,1.00) **(0.00, 0.98)**	(0.00,1.00) **(0.00, 0.98)**
1	(0.00,1.00) **(0.00, 0.94)**	(0.00,1.00) **(0.00, 0.96)**	(0.00,1.00) **(0.00, 1.00)**	(0.00,1.00) **(0.00, 0.98)**	(0.00,1.00) **(0.00, 0.98)**
1.1	(0.00,1.00) **(0.00, 0.94)**	(0.00,1.00) **(0.00, 0.96)**	(0.00,1.00) **(0.00, 1.00)**	(0.00,1.00) **(0.00, 0.98)**	(0.00,1.00) **(0.00, 0.98)**
1.2	(0.00, 1.00) **(0.00, 0.94)**	(0.00,1.00) **(0.00, 0.96)**	(0.00,1.00) **(0.00, 1.00)**	(0.00,1.00) **(0.00, 0.98)**	(0.00,1.00) **(0.00, 0.98)**
1.3	(0.00,1.00) **(0.00, 0.94)**	(0.02,1.00) **(0.06, 0.96)**	(0.17,1.00) **(0.20, 0.97)**	(0.27,1.00) **(0.30, 0.97)**	(0.36,1.00) **(0.38, 0.98)**
1.5	(0.31,1.00) **(0.37, 0.94)**	(0.49,1.00) **(0.53, 0.96)**	(0.59,1.00) **(0.62, 0.97)**	(0.66,1.00) **(0.69, 0.97)**	(0.71,1.00) **(0.73, 0.98)**
1.7	(0.58,1.00) **(0.65,0.93)**	(0.70,1.00) **(0.75, 0.95)**	(0.77,1.00) **(0.80, 0.96)**	(0.81,1.00) **(0.84,0.97)**	(0.84,1.00) **(0.86,0.98)**
1.9	(0.72,1.00) **(0.80,0.92)**	(0.81,1.00) **(0.86, 0.94)**	(0.85,1.00) **(0.90, 0.96)**	(0.88,1.00) **(0.92, 0.97)**	(0.90,1.00) **(0.93, 0.97)**
2	(0.77,1.00) **n.e.**	(0.84,1.00) **n.e.**	(0.88,1.00) **n.e.**	(0.90,1.00) **n.e.**	(0.92,1.00) **n.e.**

Note : a) The ***bold*** *figures in parentheses show the range of* w *for which the proposed estimator* $T_{\left(u,\sigma^{-1}\right)}$ *is better than MMSE estimator* $T_{\left(MMSE,\sigma^{-1}\right)}$*.b)* ***n.e*** *stands for does not exit.*

Table 4.2 PRE of $T_{\left(s,\sigma^{-1}\right)}$ with respect to unbiased estimator $T_{\left(u,\sigma^{-1}\right)}$ and MMSE estimator $T_{\left(MMSE,\sigma^{-1}\right)}$, for different values of n = 2,3,5,7,9 ; *w*= 0.25 (0.25) 0.75 and λ^*

w	λ^* ↓ n→	PRE of $T_{(s,\sigma)}$ w. r. t. $T_{\left(u,\sigma^{-1}\right)}$ 7	9	11	13	15	PRE of $T_{(s,\sigma)}$ w. r. t. $T_{\left(MMSE,\sigma^{-1}\right)}$ 7	9	11	13	15
0.25	0.3	46.42	30.78	23	18.36	15.27	41.02	28.33	21.61	17.46	14.65
	0.5	88.52	59.23	44.47	35.59	29.66	78.21	54.52	41.78	33.86	28.45
	0.7	223.87	154.37	117.71	95.1	79.76	197.8	142.08	110.6	90.47	76.51
	0.8	428.72	309.96	242.55	199.17	168.93	378.8	285.28	227.89	189.47	162.05
	0.9	950.67	784.12	666.9	580.06	513.18	839.99	721.7	626.59	551.81	492.26
	1	1600	1600	1600	1600	1600	1413.72	1472.62	1503.3	1522.09	1534.78
	1.1	950.67	784.12	666.9	580.06	513.18	839.99	721.7	626.59	551.81	492.26
	1.2	428.72	309.96	242.55	199.17	168.93	378.8	285.28	227.89	189.47	162.05
	1.3	223.87	154.37	117.71	95.1	79.76	197.8	142.08	110.6	90.47	76.51
	1.5	88.52	59.23	44.47	35.59	29.66	78.21	54.52	41.78	33.86	28.45
	1.7	46.42	30.78	23	18.36	15.27	41.02	28.33	21.61	17.46	14.65
	Range of λ^*	[0.54,1.46]	[0.63,1.37]	[0.68,1.32]	[0.71,1.29]	[0.74,1.26]	[0.57,1.43]	[0.64,1.36]	[0.69,1.31]	[0.72,1.28]	[0.75,1.25]
0.5	0.3	84.77	60.02	46.42	37.83	31.92	74.9	55.24	43.61	35.99	30.62
	0.5	138.06	102.82	81.86	67.98	58.12	121.99	94.64	76.91	64.67	55.75
	0.7	237.67	196.03	166.72	145.01	128.3	210	180.42	156.65	137.95	123.07
	0.8	306.85	273.52	246.63	224.53	206.05	271.13	251.74	231.73	213.6	197.65
	0.9	371.78	358.55	346.18	334.62	323.81	328.5	330	325.26	318.33	310.61
	1	400	400	400	400	400	353.43	368.16	375.83	380.52	383.69
	1.1	371.78	358.55	346.18	334.62	323.81	328.5	330	325.26	318.33	310.61
	1.2	306.85	273.52	246.63	224.53	206.05	271.13	251.74	231.73	213.6	197.65
	1.3	237.67	196.03	166.72	145.01	128.3	210	180.42	156.65	137.95	123.07
	1.5	138.06	102.82	81.86	67.98	58.12	121.99	94.64	76.91	64.67	55.75
	1.7	84.77	60.02	46.42	37.83	31.92	74.9	55.24	43.61	35.99	30.62
	Range of λ^*	[0.38,1.62]	[0.5,1.5]	[0.57,1.43]	[0.61,1.39]	[0.65,1.35]	[0.43,1.57]	[0.52,1.48]	[0.58,1.42]	[0.63,1.37]	[0.66,1.34]
0.75	0.3	125.8	109.1	96.28	86.15	77.93	111.15	100.42	90.46	81.95	74.76
	0.5	146.83	134.56	124.16	115.24	107.51	129.73	123.85	116.66	109.63	103.12
	0.7	165.24	159.35	153.86	148.72	143.91	146	146.67	144.56	141.48	138.05
	0.8	171.98	169.09	166.29	163.57	160.95	151.95	155.63	156.24	155.61	154.38
	0.9	176.29	175.52	174.76	174	173.25	155.77	161.55	164.2	165.53	166.19
	1	177.78	177.78	177.78	177.78	177.78	157.08	163.62	167.03	169.12	170.53
	1.1	176.29	175.52	174.76	174	173.25	155.77	161.55	164.2	165.53	166.19
	1.2	171.98	169.09	166.29	163.57	160.95	151.95	155.63	156.24	155.61	154.38
	1.3	165.24	159.35	153.86	148.72	143.91	146	146.67	144.56	141.48	138.05
	1.5	146.83	134.56	124.16	115.24	107.51	129.73	123.85	116.66	109.63	103.12
	1.7	125.8	109.1	96.28	86.15	77.93	111.15	100.42	90.46	81.95	74.76
	Range of λ^*	[0.06,1.94]	[0.24,1.76]	[0.33,1.67]	[0.42,1.58]	[0.46,1.54]	[0.18,1.2]	[0.3,1.7]	[0.38,1.62]	[0.44,1.56]	[0.49,1.51]

5. Improved classes of estimators for estimating the inverse of standard deviation (i.e. σ^{-1})

Minimizing the $MSE\left\{T_{(s,\sigma^{-1})}\right\}$, given by equation (4.3) with respect to *w* we get the optimum value of *w* as

$$w_{opt}^{(c)} = \frac{\left(1-\lambda^*\right)^2}{\left(1-\lambda^*\right)^2 + v_2(n)}$$

$$= \frac{\left(\sigma^{-1}-\sigma_0^{-1}\right)^2}{\left(\sigma^{-1}-\sigma_0^{-1}\right)^2 + \sigma^{-2}\, v_2(n)} \quad . \tag{5.1}$$

Now, replacing σ^{-1} and σ^{-2} by their unbiased estimator $T_{(u,\sigma^{-1})}$ and $T_{(u,\sigma^{-2})} = \left(\frac{n-3}{n-1}\right)s^2$ respectively in (5.1), we get a consistent estimate of $w_{opt}^{(c)}$ as:

$$\hat{w}_{opt}^{(c1)} = \frac{\left(T_{(u,\sigma^{-1})}-\sigma_0^{-1}\right)^2}{\left(T_{(u,\sigma^{-1})}-\sigma_0^{-1}\right)^2 + \left\{\frac{n-3}{n-1}\right\} v_2(n) s^{-2}} \quad . \tag{5.2}$$

Replacing σ^{-1} by its unbiased estimator $T_{(u,\sigma^{-1})}$ in (5.1), we get

$$\hat{w}_{opt}^{(c2)} = \frac{\left(T_{(u,\sigma^{-1})}-\sigma_0^{-1}\right)^2}{\left(T_{(u,\sigma^{-1})}-\sigma_0^{-1}\right)^2 + v_2(n)\,\{c_2(n)\}^2\, s^{-2}} \quad , \tag{5.3}$$

and replacing σ^{-1} by its unbiased estimator $T_{(u,\sigma^{-1})}$ and σ^{-2} by its MMSE estimator $T_{(MMSE,\sigma^{-2})} = \left(\frac{n-5}{n-1}\right)s^{-2}$ in (5.1), we get

$$\hat{w}_{opt}^{(c3)} = \frac{\left(T_{(u,\sigma^{-1})}-\sigma_0^{-1}\right)^2}{\left(T_{(u,\sigma^{-1})}-\sigma_0^{-1}\right)^2 + v_2(n)\left\{\frac{n-5}{n-1}\right\} s^{-2}} \quad . \tag{5.4}$$

Many more consistent estimators of $w_{opt}^{(c)}$ can be obtained. However, a more flexible estimator of $w_{opt}^{(c)}$ can be obtained as :

$$\hat{w}_{opt}^{(\beta)} = \frac{\left(T_{\left(u,\sigma^{-1}\right)} - \sigma_0^{-1}\right)^2}{\left(T_{\left(u,\sigma^{-1}\right)} - \sigma_0^{-1}\right)^2 + \beta_1 s^{-2}} \; , \tag{5.5}$$

where $\beta_1 (\geq 0)$ is a suitably chosen constant.

Thus, the resulting modified shrinkage estimators of the inverse of S.D. σ^{-1} are given by

$$T_{\left(s,\sigma^{-1}\right)}^{(1)} = \sigma_0^{-1} + \frac{\left(T_{\left(u,\sigma^{-1}\right)} - \sigma_0^{-1}\right)^3}{\left(T_{\left(u,\sigma^{-1}\right)} - \sigma_0^{-1}\right)^2 + \left\{\frac{n-3}{n-1}\right\} v_2(n)\, s^{-2}} \; , \tag{5.6}$$

$$T_{\left(s,\sigma^{-1}\right)}^{(2)} = \sigma_0^{-1} + \frac{\left(T_{\left(u,\sigma^{-1}\right)} - \sigma_0^{-1}\right)^3}{\left(T_{\left(u,\sigma^{-1}\right)} - \sigma_0^{-1}\right)^2 + v_2(n)\left\{c_2(n)\right\}^2 \; s^{-2}} \; , \tag{5.7}$$

$$T_{\left(s,\sigma^{-1}\right)}^{(3)} = \sigma_0^{-1} + \frac{\left(T_{\left(u,\sigma^{-1}\right)} - \sigma_0^{-1}\right)^3}{\left(T_{\left(u,\sigma^{-1}\right)} - \sigma_0^{-1}\right)^2 + \left\{\frac{n-5}{n-1}\right\} v_2(n)\; s^{-2}} \tag{5.8}$$

and

$$T_{\left(s,\sigma^{-1}\right)}^{(4)} = \sigma_0^{-1} + \frac{\left(T_{\left(u,\sigma^{-1}\right)} - \sigma_0^{-1}\right)^3}{\left\{\left(T_{\left(u,\sigma^{-1}\right)} - \sigma_0^{-1}\right)^2 + \beta_1 \; s^{-2}\right\}} \; . \tag{5.9}$$

The biases and MSE's of the estimators $T_{\left(s,\sigma^{-1}\right)}^{(i)}$, i = 1 to 4 are respectively given by

$$Bias\left\{T_{\left(s,\sigma^{-1}\right)}^{(i)}\right\} = \left(\sigma_0^{-1} - \sigma^{-1}\right) + \int_0^\infty \left[\frac{\left(T_{\left(u,\sigma^{-1}\right)} - \sigma_0^{-1}\right)^3}{\left\{\left(T_{\left(u,\sigma^{-1}\right)} - \sigma_0^{-1}\right)^2 + j' s^{-2}\right\}}\right] f(s)\, ds \tag{5.10}$$

and

$$MSE\left\{T^{(i)}_{\left(s,\sigma^{-1}\right)}\right\} = \left(\sigma_0^{-1} - \sigma^{-1}\right)^2 +$$

$$2\left(\sigma_0^{-1} - \sigma^{-1}\right) \int_0^\infty \left[\frac{\left(T_{\left(u,\sigma^{-1}\right)} - \sigma_0^{-1}\right)^3}{\left\{\left(T_{\left(u,\sigma^{-1}\right)} - \sigma_0^{-1}\right)^2 + j' s^{-2}\right\}} \right] f(s)\, ds$$

$$+ \int_0^\infty \left[\frac{\left(T_{\left(u,\sigma^{-1}\right)} - \sigma_0^{-1}\right)^6}{\left\{\left(T_{\left(u,\sigma^{-1}\right)} - \sigma_0^{-1}\right)^2 + j' s^{-2}\right\}^2} \right] f(s)\, ds \tag{5.11}$$

where $j' = v_2(n)\left(\dfrac{n-3}{n-1}\right),\ v_2(n)\left\{c_2(n)\right\}^2,\ v_2(n)\left(\dfrac{n-5}{n-1}\right),\ (\beta_1 \geq 0)$.

Now, transform the above expression in the form of λ^*, for this we are using transformation $y = \dfrac{(n-1)s^2}{2\sigma^2}$, we have $MSE\left\{T^{(i)}_{\left(s,\sigma^{-1}\right)}\right\} =$

$$\sigma^{-2} \left\{ \left(\lambda^* - 1\right)^2 + \frac{2\left(\lambda^* - 1\right)}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty \left[\frac{\left(C_6(n)\, y^{-\frac{1}{2}} - \lambda^*\right)^3}{\left\{\left(C_6(n)\, y^{-\frac{1}{2}} - \lambda^*\right)^2 + \left(\frac{n-1}{2}\right) j' \left(y\right)^{-1}\right\}} \right] e^{-y} y^{\left(\frac{n-3}{2}\right)} dy \right.$$

$$\left. + \frac{1}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty \left[\frac{\left(C_6(n)\, y^{-\frac{1}{2}} - \lambda^*\right)^6}{\left\{\left(C_6(n)\, y^{-\frac{1}{2}} - \lambda^*\right)^2 + \left(\frac{2}{n-1}\right) j' \left(y\right)^{-1}\right\}^2} \right] e^{-y} y^{\left(\frac{n-3}{2}\right)} dy \right\} \tag{5.12}$$

where $C_6(n) = \dfrac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)}$.

The PRE of $T^{(i)}_{(s,\sigma^{-1})}$ w.r.t unbiased estimator $T_{(u,\sigma^{-1})}$ and MMSE estimator $T_{(MMSE,\sigma^{-1})}$ for different values $j' = v_2(n)\left(\frac{n-3}{n-1}\right)$, $v_2(n)\{c_2(n)\}^2$ and $v_2(n)\left(\frac{n-5}{n-1}\right)$ respectively for i=1,2,3 are given in the Tables 5.1, 5.2 and 5.3. It is observed that the proposed modified estimators $T^{(i)}_{(s,\sigma^{-1})}$ are better than $T_{(u,\sigma^{-1})}$ and $T_{(MMSE,\sigma^{-1})}$ when $\lambda^* \in [0.7, 1.3] \Rightarrow \lambda \in [0.77, 1.43]$ and $7 \le n \le 15$. For fixed λ^*, the $PRE\left\{T^{(i)}_{(s,\sigma^{-1})}, T_{(u,\sigma^{-1})}\right\}$ and $PRE\left\{T^{(i)}_{(s,\sigma^{-1})}, T_{(MMSE,\sigma^{-1})}\right\}$ decrease as n increases and for fixed n these values decreases as λ^* goes away from unity where PRE's attain its maximum. We also observe that the gain in efficiency w.r.t. $T_{(MMSE,\sigma^{-1})}$ is lesser than $T_{(u,\sigma^{-1})}$. Tables 5.1, 5.2 and 5.3 show that the estimator $T^{(2)}_{(s,\sigma^{-1})}$ is the best (in the sense of having smallest MSE) among $T^{(1)}_{(s,\sigma^{-1})}, T^{(2)}_{(s,\sigma^{-1})}$ and $T^{(3)}_{(s,\sigma^{-1})}$ followed by $T^{(1)}_{(s,\sigma^{-1})}$. Thus, the modified estimator $T^{(2)}_{(s,\sigma^{-1})}$ is to be preferred in practice when λ^* moves toward unity and larger efficiency is observed when sample size n is small.

Expression (5.1) shows that the optimum value of w depends on the unknown parameter σ^{-1}. Since σ_0^{-1} is the guessed or prior value of σ^{-1}, one may replace σ^{-1} by $\dddot{\beta}\sigma_0^{-1}$, where $\dddot{\beta}$ is a positive constant. Thus, putting σ^{-1} = $\dddot{\beta}\sigma_0^{-1}$ ($\dddot{\beta} > 0$), in (5.1), we get

$$k^{(d)}_{opt} = \frac{\left(\dddot{\beta}-1\right)^2}{\left(\dddot{\beta}-1\right)^2 + v_2(n)\dddot{\beta}^2} .$$

Thus, the resulting estimator of σ^{-1} is

$$T^{(*,opt)}_{(s,\sigma^{-1})} = \frac{\left(1-\dddot{\beta}\right)^2 T_{(u,\sigma^{-1})} + v_2(n)\sigma_0^{-1}\dddot{\beta}^2}{\left(\left(1-\dddot{\beta}\right)^2 + v_2(n)\dddot{\beta}^2\right)} . \tag{5.13}$$

The bias of $T^{(*,opt)}_{(s,\sigma^{-1})}$ is given by

$$Bias\left\{T_{\left(s,\sigma^{-1}\right)}^{(*,opt)}\right\} = -\frac{\left(1-\lambda^*\right)v_2(n)\sigma^{-1}\dddot{\beta}^2}{\left\{\left(1-\dddot{\beta}\right)^2+v_2(n)\dddot{\beta}^2\right\}} \quad .$$

The MSE of $T_{(s,\sigma)}^{(*,opt)}$ is given by

$$MSE\left\{T_{(s,\sigma)}^{(*,opt)}\right\} = \frac{\sigma^{-2}v_2(n)\left\{\left(1-\dddot{\beta}\right)^4+\left(1-\lambda^*\right)^2 v_2(n)\dddot{\beta}^4\right\}}{\left\{\left(1-\dddot{\beta}\right)^2+v_2(n)\dddot{\beta}^2\right\}^2} \quad ,$$

which is smaller than the variance of the conventional unbiased estimator $T_{\left(u,\sigma^{-1}\right)}$ if

$$\frac{\left\{\left(1-\dddot{\beta}\right)^4+\left(1-\lambda^*\right)^2 v_2(n)\dddot{\beta}^4\right\}}{\left\{\left(1-\dddot{\beta}\right)^2+v_2(n)\dddot{\beta}^2\right\}^2}<1$$

$$\text{i.e. if } \dddot{\beta} \le \left\{1+\sqrt{\frac{1}{2}\left\{\left(1-\lambda^*\right)^2-v_2(n)\right\}}\right\}^{-1} \quad , \tag{5.14}$$

which shows that $\dddot{\beta}$ should be between zero and one (i.e. $0<\dddot{\beta}\le 1$) for gain in efficiency.

Table 5.1 $PRE\{T^{(1)}_{(s,\sigma^{-1})}, T_{(u,\sigma^{-1})}\}$ and $PRE\{T^{(1)}_{(s,\sigma^{-1})}, T_{(MMSE,\sigma^{-1})}\}$

$\lambda^*\downarrow$ n→	7	9	11	13	15
0.3	99.4	98.41	97.72	97.26	99.91
	92.24	**94.26**	**94.63**	**95.38**	**95.83**
0.5	103.92	97.64	94.37	93.43	93.08
	91.82	**89.87**	**88.67**	**88.88**	**89.28**
0.7	136.25	121.72	109.18	103.69	96.33
	120.38	**112.03**	**102.58**	**98.65**	**92.4**
0.8	170.34	160.61	140.94	132.5	131.37
	150.5	**147.82**	**132.42**	**126.05**	**126.02**
0.9	211.78	196.21	203.14	176.64	180.62
	187.12	**180.59**	**190.87**	**168.04**	**173.26**
1	225.12	216.16	228.07	216.34	206.5
	198.91	**198.96**	**214.28**	**205.81**	**198.08**
1.1	214.58	191.31	206.74	194.1	164.49
	189.6	**176.08**	**194.24**	**184.65**	**157.78**
1.2	194.02	156.21	152.7	167.13	129.54
	171.43	**143.77**	**143.47**	**158.99**	**124.26**
1.3	163.63	139.9	109.48	127.92	118.46
	144.58	**128.76**	**102.86**	**121.69**	**113.63**
1.5	104.83	121.61	88.44	77.53	85.6
	92.63	**111.92**	**83.1**	**73.76**	**82.12**
1.7	82.11	91.4	88.81	74.19	71.38
	72.55	**84.12**	**83.44**	**70.57**	**68.47**
Range of λ^*	[0.45 , 1.51]	[0.58 , 1.57]	[0.65 , 1.34]	[0.69 , 1.37]	[0.73 , 1.39]
	[0.6 , 1.46]	**[0.65 , 1.56]**	**[0.69 , 1.31]**	**[0.71 , 1.35]**	**[0.74 , 1.39]**

Table 5.2 $PRE\{T^{(2)}_{(s,\sigma^{-1})}, T_{(u,\sigma^{-1})}\}$ and $PRE\{T^{(2)}_{(s,\sigma^{-1})}, T_{(MMSE,\sigma^{-1})}\}$

$\lambda^*\downarrow$ n$\rightarrow$	7	9	11	13	15
0.3	99.68	95.8	95.28	99.93	99.65
	91.61	**93.7**	**94.22**	**95.06**	**95.59**
0.5	104	97.12	93.8	92.91	92.62
	91.89	**89.39**	**88.13**	**88.38**	**88.85**
0.7	139.96	123.04	109.56	103.76	96.33
	123.67	**113.25**	**102.94**	**98.71**	**92.4**
0.8	178.8	163.98	143.28	133.55	132.03
	157.98	**150.93**	**134.62**	**127.04**	**126.65**
0.9	225.48	204.02	208.44	180.73	183.19
	199.23	**187.78**	**195.85**	**171.93**	**175.73**
1	240.15	227.22	235.05	222.48	211.42
	212.19	**209.13**	**220.85**	**211.65**	**202.81**
1.1	226.45	199.72	212.19	197.5	167.43
	200.08	**183.82**	**199.36**	**187.89**	**160.61**
1.2	202.1	160.24	155.62	168.22	130.2
	178.57	**147.49**	**146.21**	**160.03**	**124.89**
1.3	168.75	141.03	110.68	128.28	118.09
	149.1	**129.8**	**103.99**	**122.03**	**113.28**
1.5	106.78	120.63	87.75	77.32	85.29
	94.35	**111.02**	**82.45**	**73.56**	**81.82**
1.7	81.9	90.85	87.56	73.37	70.85
	72.36	**83.62**	**82.27**	**69.79**	**67.96**
Range of λ^*	[0.41 , 1.53]	[0.58 , 1.60]	[0.65 , 1.34]	[0.69 , 1.37]	[0.72 , 1.41]
	[0.59 , 1.46]	**[0.65 , 1.57]**	**[0.69 , 1.31]**	**[0.71 , 1.36]**	**[0.74 , 1.39]**

Note: ***Bold*** *figures denote the PREs/ranges w.r.t. MMSE estimator*

Table 5.3 $PRE\{T^{(3)}_{(s,\sigma^{-1})}, T_{(u,\sigma^{-1})}\}$ and $PRE\{T^{(3)}_{(s,\sigma^{-1})}, T_{(MMSE,\sigma^{-1})}\}$

λ*↓ n→	7	9	11	13	15
0.3	99.1	99.34	98.13	97.10	96.95
	92.41	**95.57**	**95.83**	**96.37**	**96.64**
0.5	102.16	99.25	96.41	95.37	94.8
	90.27	**91.34**	**90.58**	**90.73**	**90.93**
0.7	119.86	115.4	107.49	103.22	96.25
	105.9	**106.21**	**100.99**	**98.2**	**92.33**
0.8	134.79	146.49	130.97	128.21	128.5
	119.09	**134.83**	**123.06**	**121.96**	**123.26**
0.9	158.34	166.19	181.79	160.51	170.33
	139.91	**152.96**	**170.81**	**152.69**	**163.38**
1	167.83	175.13	200.82	192.64	187.55
	148.29	**161.19**	**188.68**	**183.26**	**179.91**
1.1	166.88	159.09	184.95	180.27	152.89
	147.45	**146.42**	**173.77**	**171.49**	**146.66**
1.2	158.82	139.72	140.63	162.01	126.81
	140.33	**128.6**	**132.13**	**154.12**	**121.64**
1.3	140.34	134.25	104.46	126.02	119.8
	124	**123.56**	**98.15**	**119.88**	**114.92**
1.5	95.95	124.58	91.46	78.49	86.76
	84.78	**114.66**	**85.93**	**74.67**	**83.22**
1.7	84.4	93.08	93.97	77.74	73.65
	74.57	**85.67**	**88.29**	**73.96**	**70.65**
Range of λ*	[0.44 , 1.47]	[0.6 , 1.65]	[0.64 , 1.32]	[0.69 , 1.37]	[0.72 , 1.42]
	[0.65 , 1.40]	**[0.67 , 1.60]**	**[0.70 , 1.25]**	**[0.71 , 1.35]**	**[0.75 , 1.40]**

*Note: **Bold** figures denote the PREs/ranges w.r.t. MMSE estimator*

6. Simulation study

In this section we have carried out a simulation study in order to see the performance of various proposed estimators $T^{(i)}_{(s,\theta)}$ (i = 1, 2, 3) relative to unbiased and MMSE.

For simulation study a random sample of 10 observations is generated from a normal population with mean $\mu = 20$ and standard deviation $\sigma = 5$ (see, Table 6.1, using SPSS), if the prior estimate of standard deviation σ is available from the past behaviour of the population as $\sigma_0 = 4$, i.e. λ =0.8 and $\lambda^* = 1.25$.

Table 6.1 Simulated data from N(20 , 25)

22.13	16.03	15.85	17.71	25.37
24.73	23.68	11.93	25.91	22.93

The different parametric functions are estimated using their usual estimators and developed modified estimators $T_{(s,\theta)}^{(1)}, T_{(s,\theta)}^{(2)}$ and $T_{(s,\theta)}^{(3)}$, where θ may be σ and σ^{-1} and gain in efficiency with respect to conventional unbiased estimator, MMSE estimator and themselves (i.e. $T_{(s,\theta)}^{(1)}, T_{(s,\theta)}^{(2)}$ and $T_{(s,\theta)}^{(3)}$) are given in Table 6.2.

It is observed from Table 6.2 that the proposed classes of estimators are more efficient than the usual unbiased and MMSE estimators with considerable gain in efficiency.

Table 6.2 Simulation Results

Parametric Function	True Value	Estimator used	Estimate	RV or RMSE	$T_{(s,\theta)}^{(1)}$	PRE of $T_{(s,\theta)}^{(2)}$	PRE of $T_{(s,\theta)}^{(3)}$
Standard Deviation (σ)	5	Unbiased	4.9925	0.057	139.71	141.44	132.87
		MMSE	4.7232	0.0539	132.11	133.75	125.64
		$T_{(s,\theta)}^{(1)}$	4.4197	0.0408	100	101.24	95.105
		$T_{(s,\theta)}^{(2)}$	4.4063	0.0403	98.775	100	93.939
		$T_{(s,\theta)}^{(3)}$	4.4489	0.0429	105.15	106.45	100
Mean Deviation about Mean	3.9894	Unbiased	3.983	0.057	139.71	141.44	132.87
		MMSE	3.769	0.0539	132.11	133.75	125.64
		$T_{(s,\theta)}^{(1)}$	3.526	0.0408	100	101.24	95.105
		$T_{(s,\theta)}^{(2)}$	3.516	0.0403	98.775	100	93.939
		$T_{(s,\theta)}^{(3)}$	3.550	0.0429	105.15	106.45	100
**Process Capability Index	0.1	Unbiased	0.094	0.07379	126.79	129	117.88
		MMSE	0.088	0.06872	118.08	120.14	109.78
		$T_{(s,\theta)}^{(1)}$	0.107	0.0582	100	101.75	92.971
		$T_{(s,\theta)}^{(2)}$	0.107	0.0572	98.28	100	91.374
		$T_{(s,\theta)}^{(3)}$	0.104	0.0626	107.56	109.44	100
Coefficient of Variation	0.25	Unbiased	0.2496	0.057	139.71	141.44	132.87
		MMSE	0.2362	0.0539	132.11	133.75	125.64
		$T_{(s,\theta)}^{(1)}$	0.2198	0.0408	100	101.24	95.105
		$T_{(s,\theta)}^{(2)}$	0.2203	0.0403	98.775	100	93.939
		$T_{(s,\theta)}^{(3)}$	0.2224	0.0429	105.15	106.45	100

Note: *** Specification limits are set as 20 ± 1.5 so that USL=21.5 and LSL=18.5. θ can be σ or σ^{-1} as the case.*

REFERENCES

JANI, P. N. (1991). “A Class of Shrinkage Estimators for the Scale Parameter of the Experimental Distribution”, IEEE Trans. Rel. 40(1), p. 68–70.

KOUROUKLIS, S. (1994). “Estimation in the Two Parameters of the Exponential Distribution with Prior Information, IEEE. Trans. Rel. 43(3), 446–450.

PANDEY, B. N. (1979). “On Shrinkage Estimation of Normal Population Variance”, Communications in Statistics – Theory and Methods, 8, 359–365.

PANDEY, B. N. AND SINGH. J. (1977). “Estimation of the Variance of Normal Population Using Prior Information”, Journal of the Indian Statistical Association, 15, 141–150.

SAXENA, S. AND SINGH, H. P. (2004). “Estimating Various Measures in Normal Population Through a Single Class of Estimators”. Journal of the Korean Statistical Society, 33:3, pp 323–337.

SINGH, H. P. AND CHANDER, V. (2008 a). “Some Classes of Shrinkage Estimators for Estimating the Standard Deviation Towards an Interval of Normal Distribution”, Model Assisted Statistics and Application, 3, 1, 71–85.

SINGH, H. P. AND CHANDER, V. (2008 b). “A General Procedure for Estimating Various Measures of Normal Distribution Using Prior Knowledge of Exponential Distribution”, Statistics in Transition, New Series 9, (1), 139–158.

SINGH, H. P. AND CHANDER, V. (2008 c). “Estimating the Variance of an Exponential Distribution in the Presence of Large True Observations”, Austrian Journal of Statistics, 32(2), 207–216.

SINGH, H. P. AND CHANDER, V. (2008 d). “Some Classes of Shrinkage Estimators for Estimating the Scale Parameter Towards an Interval of Exponential Distribution”, Journal of Probability and Statistical Science, 6(1), 69–84.

SINGH, H. P. AND CHANDER, V. (2008 e). “Estimation of Scale Parameters Towards an Interval of Exponential Distribution”, Bulletin of Statistics and Economics (BSE), 2, A08,65–71.

SINGH, H. P. AND SAXENA, S. (2003). “An Improved Class of Shrinkage Estimators for the Variance of a Normal Population”, Statistics in Transition, 6, 119–129.

SINGH, H. P. AND SINGH, R. (1997). “A Class of Shrinkage Estimators for the Variance of a Normal Population”, Microelectronics and Reliability, 37, 863–867.

SINGH, H. P. , SHUKLA, S. K. AND KATYAR, N. P. (1999).”Estimation of Standard Deviation in Normal Distribution with Prior Information”, Proceedings of the National Academic Sciences India”, 69, 183–189.

THOMPSON, J. R. (1968). “Some Shrinkage Techniques for Estimating the Mean”, Journal of the American Statistical Association, 63, 113–122.

STATISTICS IN TRANSITION-new series, July 2009
Vol. 10, No. 1, pp. 59—73

SEARCH OF EFFECTIVE ROTATION PATTERNS IN PRESENCE OF AUXILIARY INFORMATION IN SUCCESSIVE SAMPLING OVER TWO OCCASIONS

G. N. Singh and Jaishree Prabha Karna

ABSTRACT

In successive (rotation) sampling over two occasions with partial replacement of units at current (second) occasion, utilizing the information on an auxiliary character over both the occasions along with the information from previous occasion on study character, regression type estimators for estimating the population mean at current (second) occasion have been proposed. Behaviours of the proposed estimators have been studied. Proposed estimators have been compared with the sample mean estimator when there is no matching and the optimum estimator, which is a linear combination of the means of the matched and unmatched portion of the sample at the current (second) occasion. Optimum replacement policy is also discussed. Results have been demonstrated through empirical and pictorial means of elaboration.

Key words: Successive sampling, partial replacement, partial regression coefficient, regression type, bias, mean square error, optimum replacement policy.

1. Introduction

A change is an inherent behaviour of the nature. Some types of changes directly or indirectly affect the quality of living and surrounding of human beings. This requires continuous monitoring of the real life situation in hand. If situations required to be monitored are concerned with a very large group of individuals (population or universe), it is a difficult, time taking and costly affair. To meet these requirements, rotation (successive) sampling provides a strong statistical tool for generating the reliable estimates at different occasions.

Theory of rotation (successive) sampling appears to have started with the work of Jessen (1942). He pioneered using the entire information collected in the previous investigations (occasions). Further the theory of rotation (successive) sampling was extended by Patterson (1950), Rao and Graham (1964), Gupta

(1979), Das (1982), Chaturvedi and Tripathi (1983) and many others. Sen (1971) developed estimators for the population mean on the current occasion using information on two auxiliary variates available on previous occasion. Sen (1972, 1973) extended his work for several auxiliary variates. Singh and Singh (1991) and Singh and Singh (2001) used the auxiliary information on current occasion for estimating the current population mean in two occasions successive sampling. Singh (2003) extended their work for h-occasions successive sampling. Feng and Zou (1997) and Biradar and Singh (2001) used the auxiliary information on both the occasions for estimating the current mean in successive sampling. In many situations, information on an auxiliary variates may be readily available on the first as well as on the second occasion, for example, tonnage (or seat capacity) of each vehicle or ship is known in survey sampling of transportation, number of beds in different hospitals may be known in hospital surveys, number of polluting industries is known in environmental survey, nature of employment status, educational status, food availability & medical aids of a locality are well known in advance for estimating the various demographic parameters in demographic surveys. Many other branches in biological (life) sciences could be traced out to show the benefits of the present study. Utilizing the auxiliary information on both the occasions Singh (2005), Singh and Priyanka (2006, 2007, 2008) proposed chain type ratio, difference and regression type estimators for estimating the population mean at current (second) occasion in two-occasion rotation (successive) sampling.

The objective of present work is to propose two new estimators for estimating the population mean at current (second) occasion in two-occasion rotation (successive) sampling by utilizing the information on sample partial regression coefficients. Information on sample partial regression coefficients helps us in identifying whether the variation in the study character from occasion to occasion is due to the presence of auxiliary character or the change is due to the sensitivity of the character under study itself. The detail behaviours of the proposed estimators have been studied and conclusions have been drawn with the help of tabular and pictorial representation.

2. Proposed estimators

Let $U = \left(U_1, U_2, \ldots, U_N\right)$ be a finite population of size N, which has been sampled over two occasions, and the character under study be denoted by x(y) on the first (second) occasion respectively. It is assumed that information on an auxiliary variable z (with known population mean) is available on both the occasions. A simple random sample (without replacement) of n units is taken on the first occasion. A random sub-sample of $m = n\lambda$ units is retained (matched) for its use on the second occasion, while a fresh simple random sample (without replacement) of $u = (n - m) = n\mu$ units are drawn on the second occasion from the remaining non-sampled units of the population so that the sample size on the second occasion is also n. λ and μ ($\lambda + \mu = 1$) are the fractions of matched and fresh samples respectively.

The following notations have been considered for the further use:

$\bar{X}, \bar{Y}, \bar{Z}$: The population means of the variables x, y and z respectively.

$\bar{x}_n, \bar{z}_n, \bar{y}_m, \bar{x}_m, \bar{z}_m, \bar{y}_u, \bar{z}_u$: The sample means of the respective variates of the sample sizes shown in suffices.

$\rho_{yx}, \rho_{xz}, \rho_{yz}$: The correlation coefficients between the variates shown in suffices.

S_x^2, S_y^2, S_z^2: The population mean squares of x, y and z respectively.

$\beta_{yx.z}$ and $\beta_{yz.x}$: The population partial regression coefficients.

$b_{yx.z}$ and $b_{yz.x}$: The sample partial regression coefficients.

To estimate the population mean $\bar{Y}$ on the second (current) occasion, three different estimators T_{1u}, T_{1m} and T_{2m} are considered. The estimator T_{1u} is based on the sample of size u drawn afresh on the second occasion and estimators T_{1m} and T_{2m} are based on the sample of size m common with both the occasions. Estimators T_{1u}, T_{1m} and T_{2m} are formulated as:

$$T_{1u} = \bar{y}_u + b_{yz}(u)\left(\bar{Z}-\bar{z}_u\right), \quad T_{1m} = \bar{y}_m^* + b_{yx.z}(m)\left(\bar{x}_n - \bar{x}_m\right),$$
$$T_{2m} = \bar{y}_m^* + b_{yx.z}(m)\left(\bar{x}_n^* - \bar{x}_m^*\right)$$

where

$$\bar{y}_m^* = \bar{y}_m + b_{yz.x}(m)\left(\bar{Z}-\bar{z}_m\right), \quad \bar{x}_m^* = \bar{x}_m + b_{xz}(m)\left(\bar{Z}-\bar{z}_m\right),$$
$$\bar{x}_n^* = \bar{x}_n + b_{xz}(n)\left(\bar{Z}-\bar{z}_n\right).$$

For estimating the mean on each occasion estimator, T_{1u} is an appropriate choice, while for measuring the change from occasion to occasion estimators T_{1m} and T_{2m} are the best option under the given situation. To consider both the problems simultaneously some suitable combinations of the estimators T_{1u}, T_{1m} and T_{2m} are desired.

Motivated with the above discussion two different convex linear combinations T_{11} and T_{12} of the estimators (T_{1u}, T_{1m}) and (T_{1u}, T_{2m}) have been considered for estimating the population mean $\bar{Y}$ at current (second) occasion. T_{11} and T_{12} are defined as

$$T_{11} = \varphi_1 T_{1u} + \left(1-\varphi_1\right) T_{1m} \tag{1}$$

and

$$T_{12} = \varphi_2 T_{1u} + (1-\varphi_2) T_{2m} \tag{2}$$

where φ_i (i = 1, 2) are unknown constants to be determined under certain criterion.

3. Bias and mean square error of proposed estimators T_{11} and T_{12}

Since T_{1u} is a simple linear regression estimator and T_{1m}, T_{2m} are also the simple linear regression type estimators, they are biased for population mean $\bar{Y}$. Since the final estimators T_{11} and T_{12} are convex linear combinations of (T_{1u}, T_{1m}) and (T_{1u}, T_{2m}) respectively, T_{11} and T_{12} are also the biased estimators of $\bar{Y}$. Bias B (.) and mean square error M (.) of the estimators T_{11} and T_{12} are derived up to the first order of approximations and under large sample using the following transformations:

$\bar{y}_u = (1+e_1)\bar{Y}$, $\bar{y}_m = (1+e_2)\bar{Y}$, $\bar{x}_m = (1+e_3)\bar{X}$, $\bar{x}_n = (1+e_4)\bar{X}$, $\bar{z}_m = (1+e_5)\bar{Z}$, $\bar{z}_n = (1+e_6)\bar{Z}$, $\bar{z}_u = (1+e_7)\bar{Z}$, $s_{yz}(u)=(1+e_8)S_{yz}$, $s_{xz}(m)=(1+e_9)S_{xz}$, $s_{xz}(n)=(1+e_{10})S_{xz}$, $s_z^2(u)=(1+e_{11})S_z^2$, $s_z^2(m)=(1+e_{12})S_z^2$, $s_z^2(n)=(1+e_{13})S_z^2$, $b_{yx.z}(m)=(1+e_{14})\beta_{yx.z}$, $b_{yz.x}(m)=(1+e_{15})\beta_{yz.x}$; Such that $E(e_i)=0$ and $|e_j|<1 \ \forall \ j=1,2,3,...,15$.

Under the above transformations T_{1u} and T_{jm} (j = 1, 2) take the following forms:

$$T_{1u} = (1+e_1)\bar{Y} - (e_7 + e_7 e_8 - e_7 e_{11})\beta_{yz}\bar{Z} \tag{3}$$

$$T_{1m} = \bar{Y}(1+e_2) + \beta_{yx.z}\bar{X}(e_4 - e_3 + e_4 e_{14} - e_3 e_{14}) - \beta_{yz.x}\bar{Z}(e_5 + e_5 e_{15}) \tag{4}$$

$$T_{2m} = \bar{Y}(1+e_2) - \beta_{yz.x}(e_5 + e_5 e_{15}) + \beta_{yx.z}\bar{X}(e_4 - e_3 + e_4 e_{14} - e_3 e_{14})$$

$$+\beta_{yz.x}\beta_{xz}\bar{Z}(e_5 - e_6 + e_5 e_9 - e_5 e_{12} - e_6 e_{10} + e_6 e_{13} + e_5 e_{14} - e_6 e_{14}) \tag{5}$$

Thus, we have the following theorems:

Theorem 3.1: Bias of the sequence of estimators T_{1j} (j = 1, 2) to the first order of approximations is obtained as

$$B(T_{1j}) = \varphi_j B(T_{1u}) + (1-\varphi_j) B(T_{jm}) \; ; \; (j = 1, 2) \tag{6}$$

where

$$B(T_{1u}) = \beta_{yz}\left(\frac{1}{u}-\frac{1}{N}\right)\left(\frac{\alpha_{003}}{S_z^2}-\frac{\alpha_{012}}{S_{yz}}\right) \tag{7}$$

$$B\left(T_{1m}\right) = \delta_1-\delta_2-\delta_3 \tag{8}$$

and

$$B\left(T_{2m}\right) = \delta_1-\delta_2-\delta_3+\beta_{xz}\left[\delta_5-\delta_4+\beta_{yx.z}\beta_{yz}\left(\frac{1}{m}-\frac{1}{n}\right)\left(\frac{\alpha_{003}}{S_z^2}-\frac{\alpha_{012}}{S_{yz}}\right)\right] \tag{9}$$

where

$$\delta_1 = E\left[\left(\bar{x}_n-\bar{X}\right)\left(b_{yx.z}(m)-\beta_{yx.z}\right)\right],$$

$$\delta_2 = E\left[\left(\bar{x}_m-\bar{X}\right)\left(b_{yx.z}(m)-\beta_{yx.z}\right)\right],$$

$$\delta_3 = E\left[\left(\bar{z}_n-\bar{Z}\right)\left(b_{yz.x}(m)-\beta_{yz.x}\right)\right],$$

$$\delta_4 = E\left[\left(\bar{z}_n-\bar{Z}\right)\left(b_{yx.z}(m)-\beta_{yx.z}\right)\right],$$

$$\delta_5 = E\left[\left(\bar{z}_m-\bar{Z}\right)\left(b_{yx.z}(m)-\beta_{yx.z}\right)\right],$$

and $\alpha_{rst} = E\left[\left(x-\bar{X}\right)^r\left(y-\bar{Y}\right)^s\left(z-\bar{Z}\right)^t\right]; r \geq 0, s \geq 0, t \geq 0$

Proof: The bias of the sequence of estimators T_{1j} (j = 1, 2) is given by

$$B(T_{1j}) = E\left[T_{1j}-\bar{Y}\right] = \varphi_j E(T_{1u}-\bar{Y}) + (1-\varphi_j)\, E(T_{jm}-\bar{Y}) \tag{10}$$

$$= \varphi_j B(T_{1u})+\left(1-\varphi_j\right)B(T_{jm})$$

where

$$B(T_{1u}) = E\left[T_{1u}-\bar{Y}\right] \text{ and } B(T_{jm}) = E\left[T_{jm}-\bar{Y}\right],\ \left(j = 1, 2\right)$$

Substituting the values of T_{1u}, T_{1m} and T_{2m} from equations (3)–(5) in the equation (10) and taking expectations up to $o(n^{-1})$, we have the expression for the bias of the sequence of estimators T_{1j} (j = 1, 2) as described in equation (6).

Theorem 3.2: Mean square error of the sequence of estimators T_{1j} (j = 1, 2) to the first order of approximations is obtained as

$$M(T_{1j}) = \varphi_j^2 M\left(T_{1u}\right) + \left(1-\varphi_j\right)^2 M\left(T_{jm}\right) + 2\varphi_j\left(1-\varphi_j\right)C_{1j} \quad ; (j = 1, 2) \qquad (11)$$

where

$$M(T_{1u}) = E\left(T_{1u}-\bar{Y}\right)^2 = \left(\frac{1}{u}-\frac{1}{N}\right)\left(1-\rho_{yz}^2\right) S_y^2$$

$$M(T_{1m}) = E\left(T_{1m}-\bar{Y}\right)^2 = \left[\left(\frac{1}{m}-\frac{1}{N}\right)\left(1-2\Delta_1\rho_{yz}+\Delta_1^2\right)+\left(\frac{1}{m}-\frac{1}{n}\right)\left(\Delta_2^2-2\Delta_2\rho_{yx}+2\Delta_1\Delta_2\rho_{xz}\right)\right]S_y^2$$

$$M(T_{2m}) = E\left(T_{2m}-\bar{Y}\right)^2 = \left[\left(\frac{1}{m}-\frac{1}{N}\right)\left(1-2\Delta_1\rho_{yz}+\Delta_1^2\right)+\left(\frac{1}{m}-\frac{1}{n}\right)\left(\Delta_2^2-\Delta_2^2\rho_{xz}^2-2\Delta_2\rho_{yx}\right.\right.$$

$$\left.\left.+2\Delta_2\rho_{xz}\rho_{yz}\right)\right]S_y^2$$

$$C_{11} = E\left[\left(T_{1u}-\bar{Y}\right)\left(T_{1m}-\bar{Y}\right)\right] = -\frac{S_y^2}{N}\left(1-\rho_{yz}^2\right)$$

$$C_{12} = E\left[\left(T_{1u}-\bar{Y}\right)\left(T_{2m}-\bar{Y}\right)\right] = -\frac{S_y^2}{N}\left(1-\rho_{yz}^2\right)$$

$$\text{where } \Delta_1 = \left(\frac{\rho_{yz}-\rho_{yx}\rho_{xz}}{1-\rho_{xz}^2}\right) \text{and } \Delta_2 = \left(\frac{\rho_{yx}-\rho_{yz}\rho_{xz}}{1-\rho_{xz}^2}\right)$$

Proof: It is obvious that mean square errors of the sequence of estimators T_{1j} (j = 1, 2) is given by

$$M(T_{1j}) = E\left[T_{1j}-\bar{Y}\right]^2 = E\left[\varphi_j\left(T_{1u}-\bar{Y}\right)+\left(1-\varphi_j\right)\left(T_{jm}-\bar{Y}\right)\right]^2$$

$$= \varphi_j^2 M(T_{1u}) + \left(1-\varphi_j^2\right)M\left(T_{jm}\right)+2\varphi_j\left(1-\varphi_j\right)E\left[\left(T_{1u}-\bar{Y}\right)\left(T_{jm}-\bar{Y}\right)\right] \; ; (j = 1, 2)$$

Using the expressions given in equations (2)–(5) and taking expectations up to $o(n^{-1})$, we have the expression of mean square error of the sequence of estimators T_{1j} (j = 1, 2) given in equation (11).

4. Minimum mean square error of proposed sequence of estimators T_{1j} (j = 1, 2)

Since mean square error of T_{1j} (j = 1, 2) in equation (11) is a function of unknown constant φ_j, it is minimized with respect to φ_j and subsequently the optimum values of φ_j (j = 1, 2) is obtained as

$$\varphi_{j_{opt}} = \frac{M(T_{jm})\text{-}C_{1j}}{M(T_{1u})+M(T_{jm})\text{-}2C_{1j}}; \ (j = 1, 2). \qquad (12)$$

Now substituting the value of $\varphi_{j\,opt}$ (j = 1, 2) in equation (12), we get the optimum mean square error of T_{1j} (j = 1, 2) as

$$M(T_{1j})_{opt} = \frac{M(T_{1u}).M(T_{jm})\text{-}C_{1j}^2}{M(T_{1u})+M(T_{jm})\text{-}2C_{1j}}; \ (j = 1, 2) \qquad (13)$$

Further, substituting the values of $M(T_{1u})$, $M(T_{jm})$ and C_{1j} (j = 1, 2) in equations (12) and (13), we get the simplified values of $\varphi_{j_{opt}}$ (j = 1, 2) and $M(T_{1j})_{opt}$ (j = 1, 2) , which are shown below:

$$\varphi_{1_{opt}} = \frac{\mu_1\left[A_8+\mu_1 A_7\right]}{\left[A_1+\mu_1 A_6+\mu_1^2 A_7\right]} \qquad (14)$$

$$M(T_{11})_{opt} = \frac{A_{10}+\mu_1 A_{11}+\mu_1^2 A_{12}}{A_1+\mu_1 A_6+\mu_1^2 A_7}\frac{S_y^2}{n} \qquad (15)$$

$$\varphi_{2_{opt}} = \frac{\mu_2\left[A_8+\mu_2 A_9\right]}{\left[A_1+\mu_2 A_6+\mu_2^2 A_9\right]} \qquad (16)$$

$$M(T_{12})_{opt} = \frac{A_{10}+\mu_2 A_{13}+\mu_2^2 A_{14}}{A_1+\mu_2 A_6+\mu_2^2 A_9}\frac{S_y^2}{n} \qquad (17)$$

where

$$A_1 = 1\text{-}\rho_{yz}^2, \quad A_2 = 1\text{-}2\Delta_1\rho_{yz}+\Delta_1^2, \quad A_3 = \Delta_2^2\text{-}2\Delta_2\rho_{yx}+2\Delta_1\Delta_2\rho_{xz},$$

$$A_4 = \Delta_2^2\text{-}\Delta_2^2\rho_{xz}^2\text{-}2\Delta_2\rho_{yx}+2\Delta_2\rho_{xz}\rho_{yz}, \quad A_5 = A_1\text{-}A_2, \quad A_6 = \text{-}(1\text{-}f)A_5,$$

$$A_7 = A_3\text{-}fA_5, \quad A_8 = A_2\text{+}fA_5, \quad A_9 = A_4\text{-}fA_5, \quad A_{10} = (1\text{-}f)A_1A_2,$$

$$A_{11} = (A_3\text{-}f^2A_5)A_1, \quad A_{12} = \text{-}fA_1A_7, \quad A_{13} = (A_4\text{-}f^2A_5)A_1,$$

$$A_{14} = \text{-}fA_1A_9 \quad \text{and } f = \frac{n}{N}.$$

5. Optimum replacement policy

To determine the optimum values of μ_j (j =1, 2) (fraction of samples to be taken afresh at second occasion) so that population mean $\overline{Y}$ may be estimated with maximum precision, we minimize mean square errors of T_{1j} (j =1, 2) given in equations (15) and (17) respectively with respect to μ_j (j =1, 2), which result in quadratic equations in μ_j (j =1, 2) and respective solutions of μ_j, say $\hat{\mu}_j$ (j =1, 2) are given below:

$$Q_1\mu_1^2 + 2Q_2\mu_1 + Q_3 = 0 \tag{18}$$

$$\hat{\mu}_1 = \frac{-Q_2 \pm \sqrt{Q_2^2 - Q_1Q_3}}{Q_1} \tag{19}$$

$$Q_4\mu_2^2 + 2Q_5\mu_2 + Q_6 = 0 \tag{20}$$

$$\hat{\mu}_2 = \frac{-Q_5 \pm \sqrt{Q_5^2 - Q_4Q_6}}{Q_4} \tag{21}$$

where $Q_1 = A_6A_{12}\text{-}A_7A_{11}$, $Q_2 = A_1A_{12}\text{-}A_7A_{10}$, $Q_3 = A_1A_{11}\text{-}A_6A_{10}$, $Q_4 = A_6A_{14}\text{-}A_9A_{13}$, $Q_5 = A_1A_{14}\text{-}A_9A_{10}$ and $Q_6 = A_3A_{13}\text{-}A_6A_{10}$.

From equations (18) and (20), it is obvious that real values of μ_j (j =1, 2) exist if the quantities under square roots are greater than or equal to zero. For any combination of correlations ρ_{yx}, ρ_{xz} and ρ_{yz}, which satisfy the conditions of real solutions, two real values of $\hat{\mu}_j$ (j =1, 2) are possible. Hence, while choosing the values of $\hat{\mu}_j$, it should be remembered that $0 \le \hat{\mu}_j \le 1$, all the other values of μ_j (j =1, 2) are inadmissible. Substituting the admissible values of $\hat{\mu}_j$ say $\mu_j^{(0)}$ (j =1, 2) from equations (19) and (21) into equations (15) and (17) respectively, we have the optimum values of mean square errors of T_j (j =1, 2), which are shown below:

$$M(T_{11})_{opt} = \frac{A_{10}+\mu_1^{(0)}A_{11}+\mu_1^{(0)2}A_{12}}{A_1+\mu_1^{(0)}A_6+\mu_1^{(0)2}A_7}\frac{S_y^2}{n} \tag{22}$$

$$M(T_{12})_{opt} = \frac{A_{10}+\mu_2^{(0)}A_{13}+\mu_2^{(0)2}A_{14}}{A_1+\mu_2^{(0)}A_6+\mu_2^{(0)2}A_9}\frac{S_y^2}{n} \tag{23}$$

6. Efficiency comparison

The percent relative efficiency of the sequence of estimators $T_{1j}(j = 1, 2)$ with respect to (i) $\overline{y}_n$, when there is no matching and (ii) $\hat{\overline{Y}}=\varphi^*\overline{y}_u+(1-\varphi^*)\overline{y}'_m$, when no auxiliary information is used at any occasion, where $\overline{y}'_m=\overline{y}_m+\beta_{yx}\left(\overline{x}_n-\overline{x}_m\right)$, have been obtained for different choices of ρ_{yx}, ρ_{yz} and ρ_{xz}. Since $\overline{y}_n$ and $\hat{\overline{Y}}$ are unbiased estimators of $\overline{Y}$, following Sukhatme et al. (1984), the variance of $\overline{y}_n$ and optimum variance of $\hat{\overline{Y}}$ are given by

$$V(\overline{y}_n) = \left(\frac{1}{n}-\frac{1}{N}\right)S_y^2$$

$$V\left(\hat{\overline{Y}}\right)_{opt^*} = \left[1+\sqrt{1-\rho_{yx}^2}\right]\frac{S_y^2}{2n}-\frac{S_y^2}{N}$$

For N=5000, n=500 and different choices of ρ_{yx}, ρ_{yz} and ρ_{xz}, Tables 1–2 give the optimum values of μ_j (j = 1, 2) and percent relative efficiencies $E_j^{(1)}$ and $E_j^{(2)}$ of $T_{1j}(j = 1, 2)$ with respect to $\overline{y}_n$ and $\hat{\overline{Y}}$ respectively, where

$$E_j^{(1)} = \frac{V(\overline{y}_n)}{M\left(T_{1j}^0\right)_{opt}}\times 100 \text{ and } E_j^{(2)} = \frac{V\left(\hat{\overline{Y}}\right)_{opt^*}}{M\left(T_{1j}^0\right)_{opt}}\times 100 \quad , (j = 1, 2).$$

The percent relative efficiencies are also demonstrated by means of pictorial representation for some choices of correlations.

Table 1. Optimum values of μ_1 and percent relative efficiencies of T_{11} with respect to $\bar{y}_n$ and $\hat{\bar{Y}}$

ρ_{yx}		0.3			0.5			0.7			0.9		
ρ_{xz}	ρ_{yz}	$\mu_1^{(o)}$	$E_1^{(1)}$	$E_1^{(2)}$	$\mu_1^{(o)}$	$E_1^{(1)}$	$E_1^{(2)}$	$\mu_1^{(o)}$	$E_1^{(1)}$	$E_1^{(2)}$	$\mu_1^{(o)}$	$E_1^{(1)}$	$E_1^{(2)}$
0.3	0.3	0.5471	111.42	108.57	0.5669	116.28	107.62	0.6088	126.74	106.61	0.7125	153.47	105.37
	0.5	0.5447	134.47	131.03	0.5636	140.08	129.66	0.6076	153.40	129.04	0.7287	191.41	131.42
	0.7	0.5429	196.95	191.91	0.5638	206.12	190.77	0.6236	232.78	195.81	*	-	-
	0.9	0.5415	527.01	513.53	0.5827	575.66	532.82	*	-	-	*	-	-
0.5	0.3	0.6186	110.67	107.84	0.6344	114.55	106.02	0.6713	123.71	104.06	0.7727	149.70	102.79
	0.5	0.6158	133.46	130.04	0.6267	136.69	126.52	0.6581	146.11	122.91	0.7440	172.55	118.47
	0.7	0.6160	196.35	191.32	0.6212	198.62	183.83	0.6524	212.33	178.61	0.7586	260.53	178.88
	0.9	0.6319	545.70	531.73	0.6171	528.30	488.98	0.6705	591.59	497.64	*	-	-
0.7	0.3	0.7330	110.17	107.35	0.7453	113.17	104.74	0.7802	121.81	102.46	*	-	-
	0.5	0.7323	133.46	130.04	0.7359	134.53	124.52	0.7577	141.02	118.62	0.8317	163.58	112.32
	0.7	0.7419	200.45	195.32	0.7319	196.08	181.49	0.7443	201.51	169.51	0.8000	226.22	155.32
	0.9	*	-	-	0.7447	541.37	501.07	0.7353	530.38	446.15	0.8218	633.64	435.06
0.9	0.3	0.8968	109.92	107.11	0.9060	112.19	103.84	*	-	-	*	-	-
	0.5	0.9011	134.64	131.20	0.8971	133.46	123.53	0.9114	137.73	115.86	*	-	-
	0.7	*	-	-	0.9016	198.23	183.48	0.8980	196.66	165.43	0.9313	211.30	145.08
	0.9	*	-	-	*	-	-	0.9071	538.64	453.10	0.9032	533.99	366.64

* *indicates that* $\mu_1^{(0)}$ *does not exist.*

Table 2. Optimum values of μ_2 and percent relative efficiencies of T_{12} with respect to $\bar{y}_n$ and $\hat{\bar{Y}}$

ρ_{yx}		0.3			0.5			0.7			0.9		
ρ_{xz}	ρ_{yz}	$\mu_2^{(o)}$	$E_2^{(1)}$	$E_2^{(2)}$	$\mu_2^{(o)}$	$E_2^{(1)}$	$E_2^{(2)}$	$\mu_2^{(o)}$	$E_2^{(1)}$	$E_2^{(2)}$	$\mu_2^{(o)}$	$E_2^{(1)}$	$E_2^{(2)}$
0.3	0.3	0.5505	111.25	108.41	0.5679	115.52	106.92	0.6037	124.41	104.65	0.6829	144.63	**
	0.5	0.5484	134.35	130.91	0.5650	139.30	128.93	0.6026	150.64	126.72	0.6937	178.91	122.84
	0.7	0.5468	196.86	191.82	0.5652	204.94	189.69	0.6158	227.43	191.31	0.8037	315.71	216.77
	0.9	0.5455	526.94	513.45	0.5816	569.49	527.10	0.8542	914.76	769.49	*	-	-
0.5	0.3	0.6571	110.35	107.53	0.6662	112.57	104.19	0.6853	117.29	**	0.7230	126.70	**
	0.5	0.6554	133.40	129.99	0.6618	135.30	125.23	0.6788	140.36	118.07	0.7145	151.13	103.76
	0.7	0.6555	196.24	191.22	0.6586	197.58	182.87	0.6759	205.13	172.55	0.7190	224.29	154.00
	0.9	0.6648	537.55	523.79	0.6562	527.50	488.24	0.6849	561.33	472.18	0.8713	791.85	543.69
0.7	0.3	0.9783	109.89	101.71	0.9784	109.91	**	0.9785	109.94	**	0.9783	133.33	129.92
	0.5	0.9783	133.33	123.41	0.9784	133.35	112.17	0.9785	133.38	**	0.9783	196.09	191.07
	0.7	0.9783	196.07	181.48	0.9783	196.09	164.95	0.9784	196.13	134.67	0.9786	526.60	513.12
	0.9	0.9783	526.35	487.18	0.9783	526.32	442.74	0.9785	526.50	361.50	*	-	-

** indicates that $\mu_2^{(0)}$ does not exist and '**' indicates no gain.*

Figure 1. The PRE of T_{11} w. r. t. $\overline{y}_n$ for $\rho_{xz} = 0.5$ and different values of ρ_{yz} and ρ_{yx}

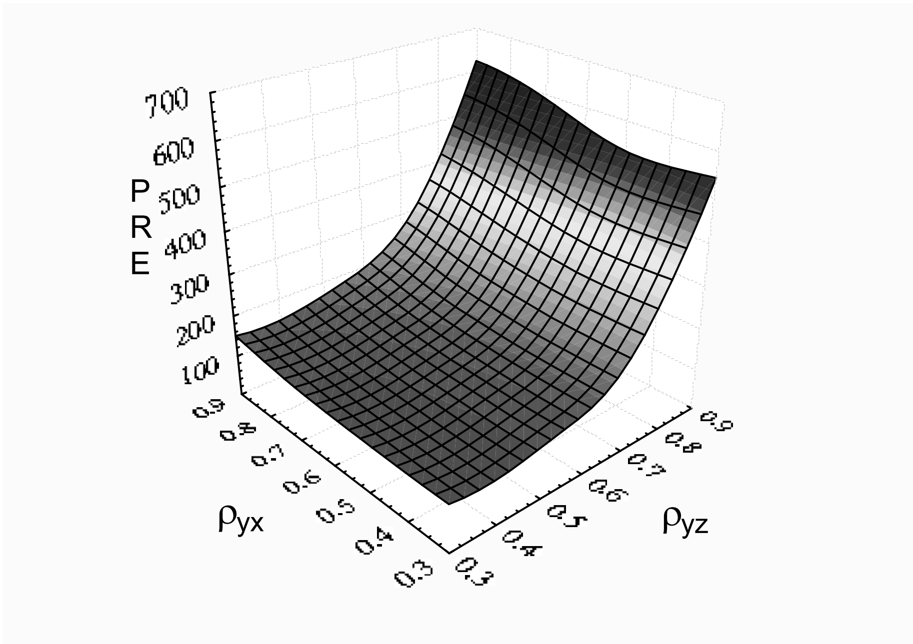
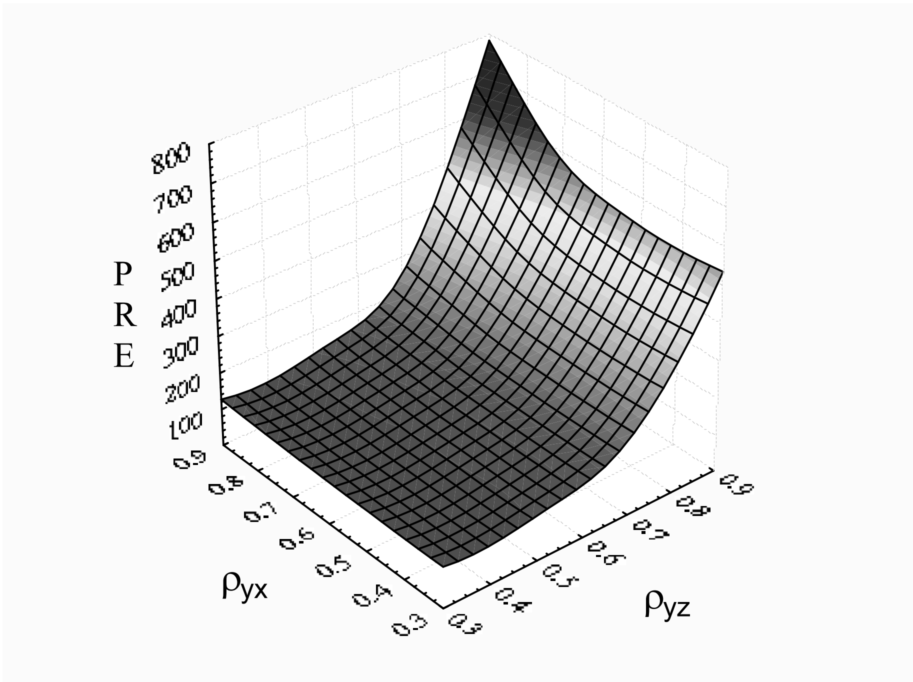


Figure 2. The PRE of T_{12} w. r. t. $\bar{y}_n$ for $\rho_{xz} = 0.5$ and different values of ρ_{yz} and ρ_{yx}



7. Conclusion

The following conclusions can be read out from Tables 1–2:

(1) From Table 1 it is noticed that

(a) For fixed values of ρ_{xz} and $\rho_{yz,}$ the values of $E_1^{(1)}$, $E_1^{(2)}$ and $\mu_1^{(0)}$ are increasing with the increasing values of $\rho_{yx.}$ This behaviour is in agreement with Sukhatme et al. (1984) results, which explains that the more the value of ρyx, more the fraction of fresh sample is required at the current occasion.

(b) For the smaller values of ρ_{xz} and $\rho_{yx,}$ the values of $E_1^{(1)}$ and $E_1^{(2)}$ are increasing while the values of $\mu_1^{(0)}$ are decreasing with the increasing values of ρ_{yz} . For the larger values of ρ_{xz} and ρ_{yx} , the values of $E_1^{(1)}$ and $E_1^{(2)}$ are increasing with the increasing values of ρ_{yz} while no definite patterns are noticed for the variations in the values of $\mu_1^{(0)}$ in similar situations. This behaviour is highly desirable in terms of percent relative efficiencies, since it concludes that if highly correlated auxiliary character is available, it pays in terms of enhance precision of estimates.

(c) For fixed values of ρ_{yz} and ρ_{yx}, the values of $E_1^{(1)}$ and $E_1^{(2)}$ are decreasing while the values of $\mu_1^{(0)}$ are increasing with the increasing values of ρ_{xz}. This indicates that behaviour of the correlation ρ_{xz} at first occasion is sensitive in terms of percent relative efficiencies and amount of fresh sample to be replaced at current occasion.

(2) From Table 2 it is observed that

(a) For fixed values of ρ_{yz} and ρ_{xz}, the values of $\mu_2^{(0)}$ are increasing and the values of $E_2^{(1)}$ are decreasing while the values of $E_2^{(2)}$ do not follow any definite pattern with the increasing values of ρ_{yx}.

(b) For fixed values of ρ_{xz} and ρ_{yx}, the values of $E_2^{(1)}$ and $E_2^{(2)}$ are increasing with the increasing values of ρ_{yz} while no definite patterns are observed in the values of $\mu_2^{(0)}$.

(c) For fixed values of ρ_{yz} and ρ_{yx}, the values of $\mu_{21}^{(0)}$ increase while $E_2^{(1)}$ and $E_2^{(2)}$ do not show any predictable pattern when the values of ρ_{xz} are increased. Nevertheless, overall gain is observed in terms of percent relative efficiencies.

Acknowledgement

Authors are grateful to the University Grants Commission, New Delhi for providing the financial assistance to carry out this research work.

REFERENCES

BIRADAR, R. S. and SINGH, H. P. (2001), Successive sampling using auxiliary information on both occasions. *Cal. Stat. Assoc. Bull., 51,No. (203–204), 243–251.*

CHATURVEDI, D. K. and TRIPATHI, T. P. (1983), Estimation of population ratio on two occasions using multivariate auxiliary information. *Jour. Ind. Statist. Assoc., 21, 113–120.*

DAS, A. K. (1982), Estimation of population ratio on two occasions. *Jour Ind. Soc. Agr. Statist., 34, 1–9.*

FENG, S. and ZOU, G. (1997), Sample rotation method with auxiliary variable. *Commun. Statist. Theo-Meth., 26, 6, 1497–1509.*

GUPTA, P. C. (1979), Sampling on two successive occasions. *Jour. Statist. Res., 13, 7–16.*

JESSEN, R. J. (1942), Statistical investigation of a sample survey for obtaining farm facts. *Iowa Agricultural Experiment Station Road Bulletin No. 304, Ames, USA, 1–104.*

PATTERSON, H. D. (1950), Sampling on successive occasions with partial replacement of units. *Jour. Royal Statist. Assoc., Scr. B, 12, 241–255.*

RAO, J. N. K. and GRAHAM, J. E. (1964), Rotation design for sampling on repeated occasions. *Jour. Amer. Statist. Assoc., 59, 492–509.*

SEN, A. R. (1971), Successive sampling with two auxiliary variables. *Sankhya, Ser. B, 33, 371–378.*

SEN, A. R. (1972), Successive sampling with p $(p \geq 1)$ auxiliary variables. *Ann. Math. Statist., 43, 2031–2034.*

SEN, A. R. (1973), Theory and application of sampling on repeated occasions with several auxiliary variables. *Biometrics, 29, 381–385.*

SINGH, V. K., and SINGH, G. N. (1991), Chain-type regression estimators with two auxiliary variables under double sampling scheme. *Merton, 49, 279–289.*

SINGH, G. N. and SINGH, V. K. (2001), On the use of auxiliary information in successive sampling. *Jour. Ind. Soc. Agri. Statist., 54(1), 1–12.*

SINGH, G. N. (2003), Estimation of population mean using auxiliary information on recent occasion in h occasions successive sampling *Statistics in Transition., 6(4), 523–532.*

SINGH, G. N. (2005), On the use of chain-type ratio estimator in successive sampling. *Statistics in Transition, 7(1), 21–26.*

SINGH, G. N. and PRIYANKA, K. (2006), On the use of chain-type ratio to difference estimSukhatme, S.ator in successive sampling. *IJAMAS; Vol. 5, No. S06, 41–49*

SINGH, G. N. and PRIYANKA, K. (2007), On the use of auxiliary information in search of good rotation patterns on successive occasions. *Bulletin of Statistics and Economics Journal, Vol. 1, No. A07, pp. 42–60.*

SINGH, G. N. and PRIYANKA, K., (2008), Search of good rotation patterns to improve the precision of the estimates at current occasion. *Communications in Statistics (Theory and Methods), 37 (30), 337–348*

SUKHATME, P. V., SUKHATME, B. V., and Asok, C. (1984), Sampling theory of surveys with applications, *Iowa State University Press, Iowa, USA, 3rd Revised Edition.*

STATISTICS IN TRANSITION-new series, July 2009
Vol. 10, No. 1, pp. 75—84

A CHAIN-TYPE ESTIMATOR FOR POPULATION VARIANCE USING TWO AUXILIARY VARIABLES IN TWO-PHASE SAMPLING

Housila P. Singh, Nidhi Mathur and Prem Chandra[1]

ABSTRACT

In this paper we have suggested a chain-type estimator of population variance by making use of a known coefficient of kurtosis of second auxiliary variable in two-phase sampling. It is shown that the proposed estimator is better than usual estimators under some realistic conditions.

Key words: Two-phase sampling, Chain ratio-type estimator, Study variable, Auxiliary variable, Mean square error.

1. Introduction

In many sample surveys studies, the information on more than single auxiliary variables correlated with study variable is either readily known or may be made known by diverting a part of the survey resources. This information is used at the estimation stage, thereby leading to a better choice of estimators. Out of many, ratio and regression methods of estimation are good examples in this context.

Consider a finite population $U = (U_1, U_2, ..., U_N)$. Let y and x be the variable under study and auxiliary variable, taking values y_i and x_i respectively for the i^{th} unit. Define

$$S_y^2 = \frac{N}{(N-1)}\sigma_y^2 \text{ with } \sigma_y^2 = \frac{1}{N}\sum_{i=1}^{N}\left(y_i - \overline{Y}\right)^2 ,$$

$$S_x^2 = \frac{N}{(N-1)}\sigma_x^2 \text{ with } \sigma_x^2 = \frac{1}{N}\sum_{i=1}^{N}\left(x_i - \overline{X}\right)^2 ,$$

[1] Department of Biostatistics, All India Institute of Medical Sciences, New Delhi, India.

where $\bar{Y}=\sum_{i=1}^{N} y_i \Big/ N$ and $\bar{X}=\sum_{i=1}^{N} x_i \Big/ N$ are the population means of y and x respectively.

For large N,

$$S_y^2 \cong \sigma_y^2 \text{ and } S_x^2 \cong \sigma_x^2 .$$

Isaki (1983) suggested a ratio estimator for S_y^2 of y assuming that the population variance S_x^2 of x is known. When the two variables y and x are closely related but no information is available on the population variance S_x^2 of x, we seek to estimate of S_y^2 of y from a sample 's', obtained through a two-phase (or double) selection. Allowing simple random sampling without replacement (SRSWOR) in each phase, the two phase sampling will be as follows:

(a) The first phase sample $s^1 \left(s^1 \subset U\right)$ of fixed size n is drawn to measure only x in order to obtain a good estimate S_x^2.

(b) Given s^1, the second-phase sample s $\left(s \subset s^1\right)$ of fixed size m is selected to measure y only.

Let $s_{ym}^2=\sum_{j\in s}\left(y_j-\bar{y}_m\right)^2 \Big/ (m-1)$, $\bar{y}_m=\sum_{j\in s} y_j \Big/ m$,

$s_{xm}^2=\sum_{j\in s}\left(x_j-\bar{x}_m\right)^2 \Big/ (m-1)$, $\bar{x}_m=\sum_{j\in s} x_j \Big/ m$, and

$s_{xn}^2=\sum_{j\in s^1}\left(x_j-\bar{x}_n\right)^2 \Big/ (n-1)$, $\bar{x}_n=\sum_{j\in s^1} x_j \Big/ n$.

The two-phase sampling version of Isaki's (1983) estimator for S_y^2 is defined by

$$t_1=s_{ym}^2\left(s_{xn}^2 / s_{xm}^2\right) \tag{1.1}$$

Sometimes even if S_x^2 is unknown, information on a cheaply ascertainable variable z is available on all units of the population with known population variance

$$S_z^2=\frac{N}{(N-1)}\sigma_z^2 \text{ with } \sigma_z^2=\frac{1}{N}\sum_{j=1}^{N}\left(z_j-\bar{Z}\right)^2 \text{ and } \bar{Z}=\sum_{j=1}^{N} z_j \Big/ N .$$

For large N, $S_z^2 \cong \sigma_z^2$.

Motivated by Chand (1975) one may suggest a chain ratio-type estimator for S_y^2 as

$$t_{R1} = s_{ym}^2 \left(\frac{s_{xn}^2}{s_{xm}^2} \right) \left(\frac{S_z^2}{s_{zn}^2} \right) \tag{1.2}$$

where $s_{zn}^2 = \sum_{j \in s^1} \left(z_j - \bar{z}_n \right)^2 \Big/ (n-1)$ with $\bar{z}_n = \sum_{j \in s^1} z_j \Big/ n$.

To the first degree of approximation, the biases of t_{Rd} and t_{R1}, the variance of s_y^2 and mean square errors (MSEs) of t_{Rd} and t_{R1} are respectively given by

$$B(t_{Rd}) = \left(\frac{1}{m} - \frac{1}{n} \right) S_y^2 \, C_1^2 \left(1 - \kappa_{01}\right) \tag{1.3}$$

$$B(t_{R1}) = S_y^2 \left[\left(\frac{1}{m} - \frac{1}{n} \right) C_1^2 \left(1 - \kappa_{01}\right) + \left(\frac{1}{n} - \frac{1}{N} \right) C_2^2 \left(1 - \kappa_{02}\right) \right] , \tag{1.4}$$

$$Var\left(s_{ym}^2\right) = S_y^4 \left(\frac{1}{m} - \frac{1}{n} \right) C_0^2 \tag{1.5}$$

$$MSE(t_{Rd}) = S_y^4 \left[\left(\frac{1}{m} - \frac{1}{n} \right) C_0^2 + \left(\frac{1}{m} - \frac{1}{n} \right) C_1^2 \left(1 - 2\kappa_{01}\right) \right] , \tag{1.6}$$

$$MSE(t_{R1}) = MSE(t_{Rd}) + S_y^4 \left(\frac{1}{n} - \frac{1}{N} \right) C_2^2 \left(1 - 2\kappa_{02}\right) , \tag{1.7}$$

where $C_0 = \sqrt{(\lambda_{400} - 1)}$, $C_1 = \sqrt{(\lambda_{040} - 1)}$, $C_2 = \sqrt{(\lambda_{004} - 1)}$,
$\rho_{01} = (\lambda_{220} - 1) \Big/ \sqrt{(\lambda_{400} - 1)(\lambda_{040} - 1)}$,
$\rho_{02} = (\lambda_{202} - 1) \Big/ \sqrt{(\lambda_{400} - 1)(\lambda_{004} - 1)}$, $\rho_{12} = (\lambda_{022} - 1) \Big/ \sqrt{(\lambda_{040} - 1)(\lambda_{004} - 1)}$,
$\kappa_{01} = \rho_{01} \, C_0 / C_1$,
$\kappa_{02} = \rho_{02} \, C_0 / C_2$, $\kappa_{12} = \rho_{12} \, C_1 / C_2$, $\kappa_{21} = \rho_{12} \, C_2 / C_1$,
$\lambda_{pqr} = \mu_{pqr} \Big/ \left\{ \mu_{200}^{p/2} \, \mu_{020}^{q/2} \, \mu_{002}^{r/2} \right\}$,
$\mu_{pqr} = (1/N) \sum_{i=1}^{N} \left(y_i - \bar{Y}\right)^p \left(x_i - \bar{X}\right)^q \left(z_i - \bar{Z}\right)^r$, (p, q, r) being non-negative integers.

In the present investigation we have suggested a modified chain ratio-type estimator for S_y^2 and discussed its properties. Further generalization of the suggested estimator are also presented.

2. Modified estimator

As noted earlier in Section 1, the information about the second auxiliary variable z is available for all the units. Thus, the values of certain parameters of the variable z can be made available. Thus, utilizing the knowledge of S_z^2 and $\beta_2(z)$ and following Upadhyaya and Singh (1999) we suggest a ratio estimator for S_x^2 as

$$t_{xR} = s_{xn}^2 \left\{ \frac{S_z^2 + \beta_2(z)}{s_{zn}^2 + \beta_2(z)} \right\} \tag{2.1}$$

where $\beta_2(z) = \lambda_{004} = \mu_{004} / \mu_{002}^2$ is the known population coefficient of kurtosis of the variable z.

Replacing s_{xn}^2 by t_{xR} in (1.7) we get a modified chain ratio-type estimator for S_y^2 as

$$t_{R2} = s_{ym}^2 \frac{t_{xR}}{s_{xm}^2}$$

$$= s_{ym}^2 \left\{ \frac{s_{xn}^2}{s_{xm}^2} \right\} \left\{ \frac{S_z^2 + \beta_2(z)}{s_{zn}^2 + \beta_2(z)} \right\} \tag{2.2}$$

To obtain the bias and MSE of t_{R2} we write

$$s_{ym}^2 = S_y^2 (1 + e_0)$$

$$s_{xm}^2 = S_x^2 (1 + e_1)$$

$$s_{xn}^2 = S_x^2 (1 + e_2)$$

$$s_{zn}^2 = S_z^2 (1 + e_3)$$

such that $E(e_i)=0\,\forall\, i=0,1,2,3,4;$ and to the first degree of approximation,

$$E(e_0^2)=\left(\frac{1}{m}-\frac{1}{N}\right)C_0^2,\quad E(e_1\,e_2)=\left(\frac{1}{n}-\frac{1}{N}\right)C_1^2,$$

$$E(e_1^2)=\left(\frac{1}{m}-\frac{1}{N}\right)C_1^2,\quad E(e_1\,e_3)=\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{12}\,C_1C_2,$$

$$E(e_2^2)=\left(\frac{1}{n}-\frac{1}{N}\right)C_1^2,\quad E(e_2\,e_3)=\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{12}\,C_1C_2,$$

$$E(e_3^2)=\left(\frac{1}{n}-\frac{1}{N}\right)C_2^2,\quad E(e_4^2)=\left(\frac{1}{m}-\frac{1}{N}\right)C_2^2,$$

$$E(e_0\,e_1)=\left(\frac{1}{m}-\frac{1}{N}\right)\rho_{01}\,C_0C_1,\quad E(e_0\,e_4)=\left(\frac{1}{m}-\frac{1}{N}\right)\rho_{02}\,C_0C_2,$$

$$E(e_0\,e_2)=\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{01}\,C_0C_1,\quad E(e_1\,e_4)=\left(\frac{1}{m}-\frac{1}{N}\right)\rho_{12}\,C_1C_2,$$

$$E(e_0\,e_3)=\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{02}\,C_0C_2,\quad E(e_2\,e_4)=\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{12}\,C_1C_2,$$

$$E(e_3\,e_4)=\left(\frac{1}{n}-\frac{1}{N}\right)C_2^2.$$

Expressing (2.2) in terms of e's we have

$$t_{R2}=S_y^2(1+e_0)(1+e_2)(1+e_1)^{-1}(1+\theta\,e_3)^{-1} \tag{2.3}$$

where $\theta=S_z^2\big/\left\{s_z^2+\beta_2(z)\right\}$.

Expanding the right hand side of (2.3) up to second power of e's we have

$$t_{R2}\cong S_y^2\left[1+e_0-e_1+e_2-\theta\,e_3-e_0e_1\right.$$

$$\left.+e_0e_2-e_1e_2-\theta\,e_0e_3-\theta\,e_2e_3+e_1^2+\theta^2e_3^2\right]$$

or

$$\left(t_{R2}-S_y^2\right)\cong S_y^2\left[e_0-e_1+e_2-\theta\,e_3-e_0e_1+e_0e_2\right.$$

$$-e_1e_2-\theta e_0e_3-\theta e_2e_3+\theta e_1e_3+e_1^2+\theta^2e_3^2\Big] \tag{2.4}$$

Taking expectation of both sides in (2.4) we get the bias of t_{R2} to the first degree of approximation as

$$B(t_{R2})=S_y^2\left[\left(\frac{1}{m}-\frac{1}{n}\right)C_1^2(1-\kappa_{01})+\left(\frac{1}{n}-\frac{1}{N}\right)\theta C_2^2(\theta-\kappa_{02})\right] \tag{2.5}$$

Squaring both sides of (2.4), retaining terms containing the powers of e's up to second degree and then taking expectations of both sides of (2.4), we get the MSE of t_{R2} as

$$MSE(t_{R2})=S_y^4\left[\left(\frac{1}{m}-\frac{1}{N}\right)C_0^2+\left(\frac{1}{m}-\frac{1}{n}\right)C_1^2(1-2\kappa_{01})\right.$$

$$\left.+\left(\frac{1}{n}-\frac{1}{N}\right)\theta C_2^2(\theta-2\kappa_{02})\right] \tag{2.6}$$

3. Efficiency comparisons

From (1.5) and (2.6) we note that $MSE(t_{R2})<Var(s_{ym}^2)$ if

$$\left[\left(\frac{1}{m}-\frac{1}{n}\right)C_1^2(1-2\kappa_{01})+\left(\frac{1}{n}-\frac{1}{N}\right)\theta C_2^2(\theta-2\kappa_{02})\right]<0$$

which is always true if

$$\kappa_{01}>\frac{1}{2}\text{ and }\kappa_{02}>\frac{\theta}{2} \tag{3.1}$$

From (1.6) and (2.6) we see that $MSE(t_{R2})<MSE(t_{Rd})$ if

$$\kappa_{02}>\frac{\theta}{2} \tag{3.2}$$

It follows from (1.7) and (2.6) that $MSE(t_{R2})<MSE(t_{R1})$ if

$$\kappa_{02}<\frac{(1+\theta)}{2} \tag{3.3}$$

Thus from (3.2) and (3.3) it is observed that the suggested estimator t_{R2} is more efficient than t_{Rd} and t_{R1} if

$$\frac{\theta}{2} < \kappa_{02} < \frac{(1+\theta)}{2} \tag{3.4}$$

From (3.1) and (3.4) we see that the estimator t_{R2} dominates the estimators s_{ym}^2, t_{Rd} and t_{R1} if (3.4) holds good along with $C > 1/2$.

4. Generalized estimators

We define the following classes of estimators for S_y^2 as

$$t_g^{(1)} = s_{ym}^2 \left\{\frac{s_{xn}^2}{s_{xm}^2}\right\}^{\alpha_1} \left\{\frac{S_z^2 + \beta_2(z)}{s_{zn}^2 + \beta_2(z)}\right\}^{\alpha_2} \tag{4.1}$$

$$t_g^{(2)} = s_{ym}^2 \left\{\frac{s_{xn}^2}{s_{xm}^2}\right\}^{\alpha_1'} \left\{\frac{S_z^2 + \beta_2(z)}{s_{zn}^2 + \beta_2(z)}\right\}^{\alpha_2'} \left\{\frac{s_{zn}^2 + \beta_2(z)}{S_z^2 + \beta_2(z)}\right\}^{\alpha_3'} \tag{4.2}$$

where $s_{zm}^2 = \sum_{j\in s}(z_j - \bar{z}_m)^2 \Big/ (m-1)$, $\bar{z}_m = \sum_{j\in s} z_j \Big/ m$, α_i's and α_i''s (i =1,2,3) are suitably chosen constants. For (α_1, α_2)=(1,1) and $(\alpha_1', \alpha_2', \alpha_3')$=(1,1,0), the estimator $t_g^{(1)}$ and $t_g^{(2)}$ respectively reduce to the estimator t_{R2} while for $\alpha_i = \alpha_i' = 0 \ \forall \ i = 1,2,3$ and $t_g^{(j)}$ $(j = 1,2)$ reduces to s_{ym}^2.

To the first degree of approximation the mean squared errors of $t_g^{(1)}$ and $t_g^{(2)}$ are respectively given by

$$MSE\left(t_g^{(1)}\right) = S_y^4\left[\left(\frac{1}{m} - \frac{1}{N}\right)C_0^2 + \left(\frac{1}{m} - \frac{1}{n}\right)\alpha_1 C_1^2(\alpha_1 - 2\kappa_{01}) + \left(\frac{1}{n} - \frac{1}{N}\right)\alpha_2\,\theta C_2^2(\alpha_2\theta - 2\kappa_{02})\right] \tag{4.3}$$

and

$$MSE\left(t_g^{(2)}\right)=S_y^4\left[\left(\frac{1}{m}-\frac{1}{N}\right)\left\{C_0^2+\alpha_3\theta C_2^2\left(\alpha_3\theta-2\kappa_{02}\right)\right\}\right.$$

$$+\left(\frac{1}{m}-\frac{1}{n}\right)\left\{\alpha_1\left(\alpha_1-2\kappa_{01}\right)-2\alpha_1\alpha_3\theta\kappa_{21}\right\}C_1^2$$

$$\left.+\left(\frac{1}{n}-\frac{1}{N}\right)\theta\left\{\alpha_2\left(\alpha_2\theta-2\kappa_{02}\right)-2\alpha_2\alpha_3\theta\right\}C_2^2\right] \tag{4.4}$$

which are respectively minimized for

$$\left.\begin{aligned}\alpha_{10}&=\kappa_{01}\\ \alpha_{20}&=\kappa_{02}/\theta\end{aligned}\right\} \tag{4.5}$$

and

$$\left.\begin{aligned}\alpha_{10}^{'}&=\frac{\left(\kappa_{01}-\kappa_{02}\,\kappa_{21}\right)}{\left(1-\kappa_{12}\,\kappa_{21}\right)}\\ \alpha_{20}^{'}&=\frac{\left(\kappa_{01}-\kappa_{02}\,\kappa_{21}\right)\kappa_{12}}{\left(1-\kappa_{12}\,\kappa_{21}\right)\theta}\\ \alpha_{10}^{'}&=\frac{\left(\kappa_{01}\kappa_{12}-\kappa_{02}\right)}{\left(1-\kappa_{12}\,\kappa_{21}\right)\theta}\end{aligned}\right\} \tag{4.6}$$

Thus the resulting minimum mean squared errors of $t_g^{(1)}$ and $t_g^{(2)}$ are respectively given by

$$\min.MSE\left(t_g^{(1)}\right)=S_y^4\,C_0^2\left[\left(\frac{1}{m}-\frac{1}{n}\right)\left(1-\rho_{01}^2\right)+\left(\frac{1}{n}-\frac{1}{N}\right)\left(1-\rho_{02}^2\right)\right] \tag{4.7}$$

and

$$\min.MSE\left(t_g^{(2)}\right)=S_y^4\,C_0^2\left[\left(\frac{1}{m}-\frac{1}{n}\right)\left(1-\rho_{0.12}^2\right)+\left(\frac{1}{n}-\frac{1}{N}\right)\left(1-\rho_{02}^2\right)\right] \tag{4.8}$$

where $\rho_{0.12}=\sqrt{\left(\rho_{01}^2+\rho_{02}^2-2\rho_{01}\rho_{02}\rho_{12}\right)/\left(1-\rho_{12}^2\right)}$.

From (1.5), (1.6), (1.7), (2.6), (4.7) and (4.8) we have

$$Var\left(s_{ym}^{2}\right)-\min.MSE\left(t_{g}^{(1)}\right)=S_{y}^{4}C_{0}^{2}\left[\left(\frac{1}{m}-\frac{1}{n}\right)\rho_{01}^{2}+\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{02}^{2}\right] \tag{4.9}$$

$$\geq 0$$

$$MSE\left(t_{Rd}\right)-\min.MSE\left(t_{g}^{(1)}\right)=S_{y}^{4}C_{0}^{2}\left[\left(\frac{1}{m}-\frac{1}{n}\right)\left(\frac{C_{1}}{C_{0}}-\rho_{01}\right)^{2}\right.$$

$$\geq 0$$

$$\left.+\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{02}^{2}\right] \tag{4.10}$$

$$MSE\left(t_{R1}\right)-\min.MSE\left(t_{g}^{(1)}\right)=S_{y}^{4}C_{0}^{2}\left[\left(\frac{1}{m}-\frac{1}{n}\right)\left(\frac{C_{1}}{C_{0}}-\rho_{01}\right)^{2}\right.$$

$$\geq 0$$

$$\left.+\left(\frac{1}{n}-\frac{1}{N}\right)\left(\frac{C_{2}}{C_{0}}-\rho_{02}\right)^{2}\right] \tag{4.11}$$

$$MSE\left(t_{R2}\right)-\min.MSE\left(t_{g}^{(1)}\right)=S_{y}^{4}C_{0}^{2}\left[\left(\frac{1}{m}-\frac{1}{n}\right)\left(\frac{C_{1}}{C_{0}}-\rho_{01}\right)^{2}\right.$$

$$\geq 0$$

$$\left.+\left(\frac{1}{n}-\frac{1}{N}\right)\left(\theta\frac{C_{2}}{C_{0}}-\rho_{02}\right)^{2}\right] \tag{4.12}$$

$$\min.MSE\left(t_{g}^{(1)}\right)-\min.MSE\left(t_{g}^{(2)}\right)=S_{y}^{4}C_{0}^{2}\left(\frac{1}{m}-\frac{1}{n}\right)\frac{\left(\rho_{01}\rho_{12}-\rho_{02}\right)}{\left(1-\rho_{12}^{2}\right)}, \tag{4.13}$$

$$\geq 0$$

Thus, it follows from (4.9), (4.10), (4.11), (4.12) and (4.13) that the suggested estimator $t_g^{(2)}$ is the best estimator among all the estimators discussed here at its optimum conditions.

REFERENCES

CHAND L.(1975): Some ratio-type estimators based on two or more auxiliary variables. *Unpublished Ph. D. dissertation, Iowa State University, Ames, Iowa.*

ISAKI, C.T. (1983): Variance estimation using auxiliary information. *Jour. Amer. Statist. Assoc.,* 78, (381), 117–123.

UPADHYAYA, L.N. and SINGH, H.P. (1999): An estimator for population variance that utilizes the kurtosis of an auxiliary variable in sample surveys. *Vikram Mathematical Journal,* 19, 14–17.

STATISTICS IN TRANSITION-new series, July 2009
Vol. 10, No. 1, pp. 85—100

AN IMPROVED VERSION OF REGRESSION RATIO ESTIMATOR WITH TWO AUXILIARY VARIABLES IN SAMPLE SURVEYS

Housila P. Singh[1], Lakshmi N. Upadhyaya[2] and Premchandra[3]

ABSTRACT

This paper considers a family of estimators of population mean $\overline{Y}$ of the variable *y* under study using information on two auxiliary variables *x* and *z* under simple random sampling without replacement (SRSWOR) scheme. The expressions for bias and mean-squared error of the proposed family of estimators are obtained up to the first degree of approximation. In addition to many, Mohanty (1967), Khare and Srivastava (1981) and Upadhyaya and Singh (1984) estimators are identified as particular members of the suggested family. Asymptotic optimum estimator (AOE) in the family is identified with its approximate mean-squared error expression.

Key words: Family of estimators, Study variate, Auxiliary variate, Bias, Mean-squared error.

1. Introduction

In sample surveys, it is a tradition to use auxiliary information for improving the precision of estimator(s) of the population parameter(s) of the variable under study. Out of many, ratio, regression and product methods of estimation are good examples in this context. The ratio method [Cochran (1940, 1942)] has been widely used when the correlation between the character under study '*y*' and the auxiliary character '*x*' is positive. If this correlation is negative, a product estimator envisaged by Robson (1957) and Murthy (1964) may be used instead of a ratio estimator. Hansen et al. (1953) suggested that the difference estimator

[1] Professor and Head, School of Studies in Statistics, Vikram University, Ujjain- 456 010. India.
[2] Professor, Department of Applied Mathematics, Indian School of Mines, Dhanbad - 826 004. India.
[3] Biostatistics & Data Management (BDM), inVentiv Clinical, GlaxoSmithKline (GSK), Consumer Health Care Research & Development Plot no. 67, Sector 32, Gurgaon, India

which was subsequently modified to give the linear regression estimator is used when the regression line of y on x does not pass through the neighbourhood of the origin. In many survey situations information on more than single auxiliary variables is available to the investigators. Several authors including Olkin (1958), Desraj (1965), Shukla (1966), Singh (1967), John (1969), Srivastava (1971), Rao and Mudholkar (1967), Sahai et al. (1980), Singh and Upadhyaya (1981), Agrawal and Panda (1993, 94), Srivastava and Jhajj (1983), Mohanty and Pattanaik (1984), and Tuteja and Bahl (1991) have presented various extensions of the ratio and product estimators to the case where multiple auxiliary variables are used to increase the precision.

Let the population consist of N identifiable sampling units, the i-th unit labelled as U_i ($i=1,2, \ldots, N$). In the general case a simple random sample of size n of the form $(y_1, x_1), (y_2, x_2), \ldots, (y_n, x_n)$ is taken without replacement scheme from the finite population U. We are interested in estimating the population mean $\overline{Y}$ of y when the population mean $\overline{X}$ of x is known. The classical ratio and regression estimators for $\overline{Y}$ are respectively given by

$$\overline{y}_R = \overline{y}\frac{\overline{X}}{\overline{x}} \tag{1.1}$$

and

$$\overline{y}_{lr} = \overline{y} + b_{yx}(\overline{X} - \overline{x}) \tag{1.2}$$

where $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$, $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, and $b_{yx} = \frac{s_{yx}}{s_x^2}$ is the sample estimate of the population regression coefficient of y on x, $s_{yx} = \frac{1}{(n-1)}\sum_{i=1}^{n}(x_i - \overline{x})^2$, $s_x^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(x_i - \overline{x})^2$.

Further, suppose that the information on two auxiliary variables x and z is available. The 'z' may be the value at some previous time when a complete census was taken, and x is another auxiliary variable which we come across when y is being measured and it is highly correlated with y. For example, let us consider that we have to estimate the total yield of paddy in a state for which we have selected blocks as our sampling units. The area under cultivation for all blocks is known for a previous year (z_i). While measuring the total yield (y_i) for blocks selected in the sample we find that the yield has been affected by rain (x_i), which is highly correlated with the yield, x being known for all the blocks, see Mohanty (1967, pp. 16–17).

Assuming that $\overline{X}$ and $\overline{Z}$, the population means of auxiliary characters x and z, are known and there is high correlation between y and x, y and z, whereas there may or may not be any correlation between x and z, Mohanty (1967) suggested a regression ratio estimator for $\overline{Y}$ as

$$\overline{y}_{RR} = \overline{y}_{lr} \frac{\overline{Z}}{\overline{z}}$$

$$= [\,\overline{y} + b_{yx}(\overline{X} - \overline{x})\,] \frac{\overline{Z}}{\overline{z}} \tag{1.3}$$

where $\overline{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$ is the sample mean of z based on n observations drawn by SRSWOR.

Motivated by Srivastava (1967), Khare and Srivastava (1981) suggested a generalization of $\overline{y}_{RR}$ at (1.3) for $\overline{Y}$ as

$$\overline{y}_{RR}^{(\alpha)} = \overline{y}_{lr} \left(\frac{\overline{z}}{\overline{Z}}\right)^{\alpha}$$

$$= [\,\overline{y} + b_{yx}(\overline{X} - \overline{x})\,]\left(\frac{\overline{z}}{\overline{Z}}\right)^{\alpha} \tag{1.4}$$

where α is a suitably chosen constant.

Further, following Srivenkataramana (1980), Upadhyaya and Singh (1984) suggested a dual to regression ratio estimator for $\overline{Y}$ as

$$\overline{y}_{RR}^{(d)} = \overline{y}_{lr} \frac{\overline{z}^{*}}{\overline{Z}}$$

$$= [\,\overline{y} + b_{yx}(\overline{X} - \overline{x})\,]\frac{\overline{z}^{*}}{\overline{Z}} \tag{1.5}$$

where $\overline{z}^{*} = \frac{(\mathrm{N}\overline{\mathrm{Z}} - n\overline{z})}{(N-n)}$.

Keeping the form of $\overline{y}_{RR}$, $\overline{y}_{RR}^{(\alpha)}$ and $\overline{y}_{RR}^{(d)}$ in view we define a family of estimators for $\overline{Y}$ as

$$\bar{y}_{RR}^{(h)} = \bar{y}_{lr}\ h(u) \tag{1.6}$$

where $u = \dfrac{\bar{z}}{\bar{Z}}$, $h(u)$ is a function of u such that $h(1) = 1$ satisfying the following regularity conditions [see, Srivastava (1971, p. 405)]:

(i) Whatever be the sample chosen, u assumes values in a bounded, closed convex subset S of one dimensional real space containing the point '1',

(ii) In S, the function $h(u)$ is continuous and bounded,

(iii) The first and second partial derivatives of $h(u)$ exist and are continuous and bounded in S.

It may easily be observed that the estimators $\bar{y}_{RR}$, $\bar{y}_{RR}^{(\alpha)}$ and $\bar{y}_{RR}^{(d)}$ and Walsh (1970), Reddy (1973, 1974) type estimator

$$\bar{y}_{RR}^{(1)} = \bar{y}_{lr}\ [1+\alpha(u-1)]^{-1},$$

Gupta (1978) type estimators

$$\bar{y}_{RR}^{(2)} = \bar{y}_{lr}\ u^{-1}\left[1+(1-\alpha)u^{-1}\right],$$

$$\bar{y}_{RR}^{(3)} = \bar{y}_{lr}\ u\left[\alpha+(1-\alpha)u\right],$$

Sahai (1979) type estimator

$$\bar{y}_{RR}^{(4)} = \bar{y}_{lr}\ \frac{[\alpha+(1-\alpha)u]}{[\alpha u+(1-\alpha)]},$$

Sahai and Ray (1980) type estimator

$$\bar{y}_{RR}^{(5)} = \bar{y}_{lr}\ \left[2-u^{\alpha}\right],$$

Vos (1980), Adhvaryu and Gupta (1983) type estimators

$$\bar{y}_{RR}^{(6)} = \bar{y}_{lr}\ \left[\alpha+(1-\alpha)u^{-1}\right],$$

$$\bar{y}_{RR}^{(7)} = \bar{y}_{lr}\ \left[\alpha+(1-\alpha)u\right],$$

Srivenkataramana and Tracy (1980, 1981) type estimators

$$\bar{y}_{RR}^{(8)} = \bar{y}_{lr}\ \left[(1+\theta^{*})-\theta^{*}u\right],$$

$$\bar{y}_{RR}^{(9)} = \bar{y}_{lr}\ \left[(1-\theta^{**})+\theta^{**}u\right],$$

Tripathi (1980), Chaubey et al. (1984) type estimator

$$\bar{y}_{RR}^{(10)} = \bar{y}_{lr}\left[(1-\alpha)+\alpha\, u\right],$$

Srivenkataramana (1980) type estimator

$$\bar{y}_{RR}^{(11)} = \bar{y}_{lr}\left[(1+g)-g\, u\right]^{-1},$$

Ray and Sahai (1980) type estimators

$$\bar{y}_{RR}^{(12)} = \bar{y}_{lr}\,\frac{(\alpha+\theta\, u)}{\left[u+(\alpha+\theta-1)\right]},$$

$$\bar{y}_{RR}^{(13)} = \bar{y}_{lr}\,\frac{(\alpha+\theta\, u)}{\left[1+(\alpha+\theta-1)u\right]},$$

Singh and Ruiz Espejo (2003) type estimator

$$\bar{y}_{RR}^{(14)} = \bar{y}_{lr}\left[\alpha\, u^{-1}+(1-\alpha)u\right],$$

Singh and Shukla (1987) type estimator

$$\bar{y}_{RR}^{(15)} = \bar{y}_{lr}\,\frac{\left[(\alpha-2)\left\{(\alpha-1)+(\alpha-3)(\alpha-4)\right\}+(n/N)(d-1)(d-4)\,u\right]}{\left[(\alpha-1)\left\{(\alpha-2)+(n/N)(\alpha-4)\right\}+(\alpha-2)(\alpha-3)(\alpha-4)\,u\right]},$$

Mohanty and Sahoo (1987) type estimators

$$\bar{y}_{RR}^{(16)} = \bar{y}_{lr}\,\frac{u}{\left[(1-\alpha)u+\alpha\right]},$$

$$\bar{y}_{RR}^{(17)} = \bar{y}_{lr}\left[(1-\alpha)u^{2}+\alpha\, u\right]^{-1},$$

Naik and Gupta (1991 a) type estimators

$$\bar{y}_{RR}^{(18)} = \bar{y}_{lr}\left[(1+\alpha)-\alpha\, u\right],$$

$$\bar{y}_{RR}^{(19)} = \bar{y}_{lr}\,\frac{u}{\left[1+(1+\alpha)(u-1)\right]},$$

Naik and Gupta (1991 b) type estimator

$$\bar{y}_{RR}^{(20)} = \bar{y}_{lr}\left[\frac{\left\{1+a(u-1)\right\}}{\left\{1+b(u-1)\right\}}\right]^{\alpha},$$

Diana (1993) type estimator

$$\bar{y}_{RR}^{(21)} = \bar{y}_{lr} \left[u^{\delta} \left\{ a + (1-a)\, u^{\phi} \right\}^{\alpha} \right],$$

Sisodia and Dwivedi (1981) type estimator

$$\bar{y}_{RR}^{(22)} = \bar{y}_{lr} \left[b^{*} u + (1 - b^{*}) \right]^{-1},$$

Bansal and Singh (1992–93) type estimator

$$\bar{y}_{RR}^{(23)} = \bar{y}_{lr} \left[b^{*} u + (1 - b^{*}) \right]^{-\alpha},$$

Upadhyaya and Singh (1999) type estimators

$$\bar{y}_{RR}^{(24)} = \bar{y}_{lr} \left[c\, u + (1 - c) \right]^{-1},$$

$$\bar{y}_{RR}^{(25)} = \bar{y}_{lr} \left[c_1\, u + (1 - c_1) \right]^{-1},$$

$$\bar{y}_{RR}^{(26)} = \bar{y}_{lr} \left[c\, u + (1 - c) \right],$$

$$\bar{y}_{RR}^{(27)} = \bar{y}_{lr} \left[c_1\, u + (1 - c_1) \right],$$

Mohanty and Sahoo (1995) type estimators

$$\bar{y}_{RR}^{(28)} = \bar{y}_{lr} \left[d^{*} u + (1 - d^{*}) \right]^{-1},$$

$$\bar{y}_{RR}^{(29)} = \bar{y}_{lr} \left[d^{*} u + (1 - d^{*}) \right]^{-\alpha},$$

$$\bar{y}_{RR}^{(30)} = \bar{y}_{lr} \left[d_1\, u + (1 - d_1) \right]^{-1},$$

$$\bar{y}_{RR}^{(31)} = \bar{y}_{lr} \left[d_1\, u + (1 - d_1) \right]^{-\alpha},$$

etc. all being equal to $\bar{y}_{lr}$ at $\bar{z} = \bar{Z}$, belong to the proposed family $\bar{y}_{RR}^{(h)}$, where $b^{*} = \dfrac{\bar{Z}}{\bar{Z} + C_z}$, $c = \dfrac{\bar{Z}\beta_2(z)}{\bar{Z}\beta_2(z) + C_z}$, $c_1 = \dfrac{\bar{Z}C_z}{\bar{Z}C_z + \beta_2(z)}$, $d^{*} = \dfrac{\bar{Z}}{\bar{Z} + z_m}$, $d_1 = \dfrac{\bar{Z}}{\bar{Z} + z_M}$, $g = \dfrac{n}{(N-n)}$, $\theta^{*} = \dfrac{\bar{Z}}{(\alpha - \bar{Z})}$, $\theta^{**} = \dfrac{\bar{Z}}{(\alpha + \bar{Z})}$, $0 \le \theta \le 1$, C_z and $\beta_2(z)$ are known coefficients of variation and kurtosis of z, z_m and z_M are minimum and maximum *z*-values respectively , and $(\alpha, a, b, \delta, \phi)$ are suitably chosen constants.

2. Properties of the proposed family $\bar{y}_{RR}^{(h)}$

Under the usual assumptions, by Taylor's expansion method, it is observed that the bias of $\bar{y}_{RR}^{(h)}$ is of $o(n^{-1})$. Denoting the first order partial derivative of $h(u)$ with respect to u, at the point u=1, by $h_1(1)$, we obtain the variance of $\bar{y}_{RR}^{(h)}$ to $o(n^{-1})$, as

$$V(\bar{y}_{RR}^{(h)}) = V(\bar{y}_{lr}) + \frac{(N-n)}{n(N-1)} \bar{Y}^2 C_z^2 \left(h_1(1)\right)^2 \left[1 + 2\frac{C_y}{C_z}\left(\frac{\rho_{yz} - \rho_{yx}\rho_{xz}}{h_1(1)}\right)\right], \quad (2.1)$$

where

$$V(\bar{y}_{lr}) = \frac{(N-n)}{n(N-1)} \bar{Y}^2 C_y^2 (1-\rho_{yx}^2) \quad (2.2)$$

is the variance of usual linear regression estimator $\bar{y}_{lr}$ to $o(n^{-1})$, ρ_{yx}, ρ_{yz} and ρ_{xz} are the correlation coefficients between (y,x), (y,z) and (x,z) respectively, C_y and C_z are the coefficients of variation of *y* and *z* respectively.

The value of $h_1(1)$ that minimizes $V(\bar{y}_{RR}^{(h)})$ is

$$h_{1(\text{opt})} = -(\rho_{yz} - \rho_{yx}\rho_{xz})C_y / C_x \quad (2.3)$$

Thus the resulting (optimal) variance of $V\left(\bar{y}_{RR}^{(h)}\right)$ is given by

$$V(\bar{y}_{RR}^{(h)}) = V(\bar{y}_{lr}) - \frac{(N-n)}{n(N-1)} \bar{Y}^2 C_y^2 (\rho_{yz} - \rho_{yx}\rho_{xz})^2$$

$$= \frac{(N-n)}{n(N-1)} \bar{Y}^2 C_y^2 (1-\rho_{yx}^2) \left[1 - \frac{(\rho_{yz} - \rho_{yx}\rho_{xz})^2}{(1-\rho_{yx}^2)}\right] \quad (2.4)$$

Thus, we have the following theorem.

Theorem 2.1: Up to terms of order n^{-1},

$$V(\overline{y}_{RR}^{(h)}) \geq \frac{(N-n)}{n(N-1)}\overline{Y}^2 C_y^2(1-\rho_{yx}^2)\left[1-\frac{(\rho_{yz}-\rho_{yx}\rho_{xz})^2}{(1-\rho_{yx}^2)}\right]$$

with equality sign holding if

$$h_1(1) = (\rho_{yz}-\rho_{yx}\rho_{xz})(C_y / C_x).$$

It should be noted that any estimator attaining the variance (2.4) may be termed a minimum variance bound (MVB) estimator. Obviously, the MVB estimator for the family is not unique.

The variance of any estimator belonging to the proposed family $\overline{y}_{RR}^{(h)}$ can be obtained easily just by putting the proper value of $h_1(1)$ in (2.1). For example, putting $h_1(1) = -1$, α, $-g$ in (2.1) the variances of $\overline{y}_{RR}$, $\overline{y}_{RR}^{(\alpha)}$ and $\overline{y}_{RR}^{(d)}$ defined at (1.3), (1.4) and (1.5) respectively can be obtained.

We note that the family $\overline{y}_{RR}^{(h)}$ at (1.6) does not include even simple difference type estimators such as

$$\overline{y}_{RR(1)} = \overline{y}_{lr} - \alpha(u-1). \tag{2.5}$$

However, it is easily shown that if we consider a family of estimators wider than (1.6), defined by

$$\overline{y}_{RR}^{(H)} = H(\overline{y}_{lr}, u) \tag{2.6}$$

where $H(\overline{y}_{lr}, u)$ is a function of $\overline{y}_{lr}$ and u such that

$$\left.\begin{aligned} H(\overline{y}_{lr}, u)\Big|_{(\overline{Y},\overline{X},1)} &= H(\overline{Y}, 1) = \overline{Y} \\ \Rightarrow H_1(\overline{Y},1) = \frac{\partial H(.)}{\partial \overline{y}_{lr}}\Bigg|_{(\overline{Y},\overline{X},1)} &= 1 \end{aligned}\right\} \tag{2.7}$$

the minimum asymptotic variance of $\overline{y}_{RR}^{(H)}$ is equal to (2.4) and is not reduced. The estimator $\overline{y}_{RR(1)}$ is a member of the family (2.6) and attain the optimal (minimum) variance (2.4) for optimum value of α in $\bar{y}_{RR(1)}$, which can be obtained by (2.3).

3. Another family of estimators

As it is noted earlier that the information on variable z is available for all the units, thus letting $u = \frac{\bar{z}}{\bar{Z}}$, $v = \frac{s_z^2}{S_z^2}$, $s_z^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(z_i - \bar{Z})^2$, and $S_z^2 = \frac{\sum_{i=1}^{n}(z_i - \bar{Z})^2}{(N-1)}$, we define another family of estimators for $\bar{Y}$ as

$$\bar{y}_{RR}^{(g)} = \bar{y}_{lr}\ g(u,v) \tag{3.1}$$

where $g(u,v)$ is a function of (u,v) such that

$$g(1,1) = 1 \tag{3.2}$$

and it satisfies the following conditions [see, Srivastava and Jhajj (1981, p. 342)]:

1. The function $g(u,v)$ is continuous and bounded in Q.
2. The first and second order partial derivatives of $g(u,v)$ are continuous and bounded in Q.

Expanding the function $g(u,v)$ about the point (1,1) in a second order Taylor's series and noting that the second order partial derivatives are bounded, we have

$$\mathrm{E}\left(\bar{y}_{RR}^{(g)}\right) = \bar{Y} + 0(n^{-1})$$

and so the bias of $\bar{y}_{RR}^{(g)}$ is of the order n^{-1}.

We write

$$C_y^2 = \frac{\mu_{020}}{\bar{Y}^2},\ C_x^2 = \frac{\mu_{200}}{\bar{X}^2},\ C_z^2 = \frac{\mu_{002}}{\bar{Z}^2},\ \rho_{yx} = \frac{\mu_{110}}{\sqrt{\mu_{200}\mu_{020}}},\ \rho_{yz} = \frac{\mu_{011}}{\sqrt{\mu_{020}\mu_{002}}},$$

$$\rho_{xz} = \frac{\mu_{101}}{\sqrt{\mu_{020}\mu_{002}}},\ \lambda_{102} = \frac{\mu_{102}}{\mu_{002}\sqrt{\mu_{200}}},\ \lambda_{012} = \frac{\mu_{012}}{\mu_{002}\sqrt{\mu_{020}}},\ \gamma_1(z) = \frac{\mu_{003}}{\mu_{002}^{3/2}},$$

$\beta_1(z)=\gamma_1^2(z)$, $\beta_2(z)=\dfrac{\mu_{004}}{\mu_{002}^2}$, $\lambda_{rst}=\dfrac{\mu_{rst}}{\left(\mu_{200}^{r/2}\ \mu_{020}^{s/2}\ \mu_{002}^{t/2}\right)}$,

$\mu_{rst}=\dfrac{\sum_{i=1}^{N}(x_i-\overline{X})^r(y_i-\overline{Y})^s(x_i-\overline{Z})^t}{N}$, (r,s,t) being non-negative integers.

To the first degree of approximation, the variance of $\overline{y}_{RR}^{(g)}$ is given by

$$V(\overline{y}_{RR}^{(g)})=V(\overline{y}_{lr})+\frac{(N-n)}{n(N-1)}\overline{Y}^2\Big[C_z^2g_1^2(1,1)+\{\beta_2(z)-1\}g_2^2(1,1)$$

$$+2\gamma_1(z)C_zg_1(1,1)g_2(1,1)+2C_yC_z(\rho_{yz}-\rho_{yx}\rho_{xz})g_1(1,1)$$

$$+2C_y(\lambda_{012}-\rho_{yx}\lambda_{102})g_2(1,1)\Big], \tag{3.3}$$

where $g_1(1,1)$ and $g_2(1,1)$ denote the first order partial derivatives of the function $g(u,v)$ with respect to u and v respectively about the point (1,1); $g_{(1)}(1,1)$ is given in (2.2).

The variance $V(\overline{y}_{RR}^{(g)})$ at (3.3) is minimized for

$$\left.\begin{aligned}
g_{1(\text{opt})}(1,1)&=\frac{\left[\gamma_1(z)(\lambda_{012}-\rho_{yx}\lambda_{102})-(\beta_2(z)-1)(\rho_{yz}-\rho_{yx}\rho_{xz})\right]C_y}{(\beta_2(z)-\beta_1(z)-1)C_z}\\
g_{2(\text{opt})}(1,1)&=\frac{\left[\gamma_1(z)(\rho_{yz}-\rho_{yx}\rho_{xz})-(\lambda_{012}-\rho_{yx}\lambda_{102})\right]C_y}{(\beta_2(z)-\beta_1(z)-1)}
\end{aligned}\right\} \tag{3.4}$$

Thus the resulting (optimal) variance of $\overline{y}_{RR}^{(g)}$ is given by

$$V_{\text{opt}}(\overline{y}_{RR}^{(g)})$$

$$=\frac{(N-n)}{(N-1)n}\overline{Y}^2C_y^2(1-\rho_{yx}^2)\left[1-\frac{(\rho_{yz}-\rho_{yx}\rho_{xz})^2}{(1-\rho_{yx}^2)}\right.$$

$$\left.-\frac{D^2}{(1-\rho_{yx}^2)(\beta_2(z)-\beta_1(z)-1)}\right] \tag{3.5}$$

where $D = \left[\gamma_1(z)(\rho_{yz} - \rho_{yx}\rho_{xz}) - (\lambda_{012} - \rho_{yx}\lambda_{102})\right]$.

In (3.5), the first term on the right-hand side gives the approximate variance of the usual linear regression estimator $\bar{y}_{lr}$ in which $\bar{X}$ is used and the first two terms give the minimum (optimal) asymptotic variance of the family $\bar{y}_{RR}^{(h)}$ at (1.6) where both $\bar{X}$ and $\bar{Z}$ are used. The third term thus gives the reduction in asymptotic variance when S_z^2 is also used along with $\bar{X}$ and $\bar{Z}$.

Any parametric function $g(u,v)$ such that $g(1,1)$=1 and satisfies the conditions 1 and 2 can generate an estimator of the family $\bar{y}_{RR}^{(g)}$ at (3.1). The family of such estimators is very large. The following functions, for example, give some simple estimators of the family:

$$g(u,v) = u^{\alpha}\, v^{\beta},\; g(u,v) = w_1 u^{\alpha} + w_2 v^{\beta};\; w_1 + w_2 = 1,$$

$$g(u,v) = \left[1 + \alpha(u-1) + \beta(v-1)\right], g(u,v) = \frac{\left[1 + \alpha(u-1)\right]}{\left[1 + \beta(v-1)\right]},$$

$$g(u,v) = \left[1 - \alpha(u-1) - \beta(v-1)\right]^{-1} g(u,v) = \exp\left[\alpha(\mathrm{u}-1) + \beta(\mathrm{v}-1)\right].$$

The optimum values of α and β in the estimators defined by above functions are obtained from the conditions (3.4) and with these optimum values, the asymptotic minimum variance is given by (3.5).

We also state the following theorem:

Theorem 3.1: Up to terms of order n^{-1},

$$V(\bar{y}_{RR}^{(g)})$$

$$\geq \frac{(N-n)}{(N-1)n}\bar{Y}^2 C_y^2 (1-\rho_{yx}^2)\left[1 - \frac{(\rho_{yz} - \rho_{yx}\rho_{xz})^2}{(1-\rho_{yx}^2)} - \frac{D^2}{(1-\rho_{yx}^2)\left(\beta_2(z) - \beta_1(z) - 1\right)}\right]$$

with equality sign holding if

$$g_1(1,1) = g_{1(\mathrm{opt})}(1,1)$$
$$g_2(1,1) = g_{2(\mathrm{opt})}(1,1)$$

where $g_{1(\mathrm{opt})}(1,1)$ and $g_{2(\mathrm{opt})}(1,1)$ are defined in (3.4).

It can be easily shown that if we consider a wider family of estimators

$$\bar{y}_{RR}^{(G)} = G(\bar{y}_{lr}, u, v) \tag{3.6}$$

of $\bar{Y}$, where the function $G(.)$ satisfies $G(\bar{Y}, 1, 1) = \bar{Y}$ and $G_1(\bar{Y}, 1, 1) = 1$, $G_1(.)$ denoting the first partial derivative of $G(.)$ with respect to $\bar{y}_{lr}$, the optimal variance of $\bar{y}_{RR}^{(G)}$ is equal to (3.5) and is not reduced. The difference-type estimator

$$\bar{y}_{RR1}^{(G)} = \bar{y}_{lr} + \alpha(u-1) + \beta(v-1)$$

is a member of the family (3.6) but not that of the family defined in (3.1).

4. Efficiency Comparisons

In order to compare the estimators considered by Mohanty (1967), Khare and Srivastava (1981) as well as Singh and Upadhyaya (1984) with the proposed family of estimators, we write the variances of their estimators up to terms of order n^{-1} as

$$V(\bar{y}_{RR}) = V(\bar{y}_{lr}) + \frac{(N-n)}{n(N-1)}\bar{Y}^2 C_z^2\left[1 - 2(\rho_{yz} - \rho_{yx}\rho_{xz})(C_y / C_z)\right] \tag{4.1}$$

$$V_{\text{opt}}(\bar{y}_{RR}^{(\alpha)}) = V_{\text{opt}}(\bar{y}_{RR}^{(h)}) \tag{4.2}$$

$$V_{\text{opt}}(\bar{y}_{RR}^{(d)}) = V(\bar{y}_{lr})$$

$$+\frac{(N-n)}{n(N-1)}\bar{Y}^2 C_z^2 g^2\left[1 - (2/g)(\rho_{yz} - \rho_{yx}\rho_{xz})(C_y / C_z)\right] \tag{4.3}$$

From (2.2), (2.4), (3.5), (4.1), (4.2) and (4.3), we have

$$V(\bar{y}_{lr}) - V_{\text{opt}}(\bar{y}_{RR}^{(h)}) = \frac{(N-n)}{n(N-1)}\bar{Y}^2 C_y^2 (\rho_{yz} - \rho_{yx}\rho_{xz})^2 \geq 0 \tag{4.4}$$

$$V(\bar{y}_{RR}) - V_{\text{opt}}(\bar{y}_{RR}^{(h)}) = \frac{(N-n)}{n(N-1)}\bar{Y}^2 \left[C_y(\rho_{yz} - \rho_{yx}\rho_{xz}) - C_z\right]^2 > 0 \tag{4.5}$$

$$V(\bar{y}_{RR}^{(d)}) - V_{\text{opt}}(\bar{y}_{RR}^{(h)}) = \frac{(N-n)}{n(N-1)}\bar{Y}^2 \left[C_y(\rho_{yz} - \rho_{yx}\rho_{xz}) - gC_z\right]^2 > 0 \tag{4.6}$$

$$V(\overline{y}_{RR}^{(h)}) - V_{\text{opt}}(\overline{y}_{RR}^{(g)}) = \frac{(N-n)}{n(N-1)}\overline{Y}^2\, C_y^2\, \frac{D^2}{\left(\beta_2(z)-\beta_1(z)-1\right)} > 0 \tag{4.7}$$

From (4.4) to (4.7), we have the following inequalities:

$$V_{\text{opt}}(\overline{y}_{RR}^{(g)}) < V_{\text{opt}}(\overline{y}_{RR}^{(h)}) < V(\overline{y}_{lr}) \tag{4.8}$$

$$V_{\text{opt}}(\overline{y}_{RR}^{(g)}) < V_{\text{opt}}(\overline{y}_{RR}^{(h)}) < V(\overline{y}_{RR}) \tag{4.9}$$

and

$$V_{\text{opt}}(\overline{y}_{RR}^{(g)}) < V_{\text{opt}}(\overline{y}_{RR}^{(h)}) < V(\overline{y}_{RR}^{(d)}) \tag{4.10}$$

Thus, we find that the proposed family $\overline{y}_{RR}^{(g)}$ at (3.1) is the best (in the sense of having least variance) among the estimators discussed here at its optimum conditions.

Acknowledgement

The authors are greatly indebted to the referee for his valuable suggestions on the paper.

REFERENCES

ADHVARYU, D. and GUPTA, P.C. (1983): On some alternative sampling strategies using auxiliary information. Metrika, 30, 217–226.

AGRAWAL, M.C. and PANDA, K.B. (1993): Multivariate product estimators, Journal of Indian Society of Agricultural Statistics, 45, 359–371.

AGRAWAL, M.C. and PANDA, K.B. (1994): On multivariate ratio estimation. Journal of Indian Statistical Association, 32, 103–110.

BANSAL, M.L. and SINGH, S. (1992–93): An improved estimator of population mean using power transformation. The Aligarh Journal of Statistics, 12, 21–24.

CHAUBEY, Y.P., SINGH, M. and DWIVEDI, T.D. (1984): A note on an optimality property of the regression estimator. Biometrical Journal, 26(4), 465–467.

COCHRAN, W.G. (1940): The estimation of the yields of cereal experiments by sampling for the ratio gain to total produce. Journal of Agricultural Society, 30, 262–275.

COCHRAN, W.G. (1942): Sampling theory when the sampling units are of unequal sizes. Journal of American Statistical Association, 37, 119–132.

DESRAJ (1965): On a method of using multi-auxiliary information in sample surveys. Journal of American Statistical Association, 60, 270–277.

DIANA, G. (1993): A class of estimators of the population mean in stratified random sampling. Statistica, LIII, 1, 59–68.

GUPTA, P.C. (1978): On some quadratic and higher degree ratio and product estimators. Journal of Indian Society of Agricultural Statistics, 30(2), 71–80.

HANSEN, M.H., HURWITZ, W.N. and MADOW, W.G. (1953): Sample survey methods and theory. John Wiley and Sons, New York.

JOHN, S. (1969): On multivariate ratio and product estimators. Biometrika, 56,533–536.

KHARE, B.B. and SRIVASTAVA, S.R. (1981): A generalized regression ratio estimator for the population mean using two auxiliary variables, The Aligarh Journal of Statistics, 1 (1), 43–51.

MOHANTY, S. (1967): Combination of regression and ratio estimate. Journal of Indian Statistical Association, 5, 16–29.

MOHANTY, S. and PATTANAIK, L.M. (1984): Alternative multivariate ratio estimators using geometric and harmonic means. Journal of Indian Society of Agricultural Statistics, 36, 110–118.

MOHANTY, S. and SAHOO, L.N. (1987): A class of estimators based on mean per unit and ratio estimators. Statistica, XLVII, (3), 473–477.

MOHANTY, S. and SAHOO, J. (1995): A note on improving the ratio method of estimation through linear transformation using certain known population parameters. Sankhya, 57, B, 93–102.

MURTHY, M.N. (1964): Product method of estimation. Sankhya, 26, A, 294–307.

NAIK, V.D. and GUPTA, P.C. (1991 a): A comparative study among the estimators obtained from weighted (arithmetic, geometric and harmonic) means of mean per unit estimator and ratio/product estimator. Metron, XLIX, 1–4, 233–244.

NAIK, V.D. and GUPTA, P.C. (1991 b): A general class of estimators for estimating population mean using auxiliary information. Metrika, 38, 11–17.

OLKIN, I. (1958): Multivariate ratio estimation from finite population. Biometrika, 45, 154–165.

RAO, J.N.K. and MUDHOLKAR, G.S. (1967): Generalized multivariate estimators for the mean of a finite population. Journal of American Statistical Association, 62, 1009–1012.

RAY, S.K. and SAHAI, A. (1980): Efficient families of ratio and product type estimators. Biometrika, 67, 215–217.

REDDY, V.N. (1973): On ratio and product methods of estimation. Sankhya, 35, B, 307–316.

REDDY, V.N. (1974): On transformed ratio method of estimation. Sankhya, 36, C, 59–70.

ROBSON, D.S. (1957): Applications of multivariate polykays to the theory of unbiased ratio-type estimation. Journal of American Statistical Association, 52, 511–522.

SAHAI, A. (1979): An efficient variant of the product and ratio estimator. Statistica Neerlandica, 33, 27–35.

SAHAI, A. and RAY, S.K. (1980): An efficient estimator using auxiliary information. Metrika, 27, 271–275.

SAHAI, A., CHANDER, R. and MATHUR, A.K. (1980): An alternative multivariate product estimator. Journal of Indian Society of Agricultural Statistics, 32, 2, 6–12.

SHUKLA, G.K., (1966): An alternative multivariate ratio estimate for finite population. Calcutta Statistical Association Bulletin, 15, 127–134.

SINGH, M.P. (1967): Multivariate product method of estimation for finite population. Journal of Indian Society of Agricultural Statistics, 19, 2, 1–10.

SINGH, H. P. and RUIZ ESPEJO, M. (2003): On linear regression and ratio-product estimation of a finite population mean. The Statistician, 52, (1), 59–67.

SINGH, H.P. and UPADHYAYA, L.N. (1981): Use of several auxiliary variables in dual to ratio method of estimation. Mathematical Forum, 4(2), 19–26.

SINGH, V.K. and SHUKLA, D. (1987): One parameter family of factor type ratio estimators. Metron, XLV, 1–2, 273–283.

SISODIA, B.V.S. and DWIVEDI, V.K. (1981): A modified ratio estimator using coefficient of variation of auxiliary variable. Journal of Indian Society of Agricultural Statistics, 33, 2, 13–18.

SRIVASTAVA, S.K. (1971): A generalized estimator for the mean of a finite population using multi-auxiliary information. Journal of American Statistical Association, 66, 404–407.

SRIVASTAVA, S.K. and JHAJJ, H.S. (1981): A class of estimators of the population mean in survey sampling using auxiliary information. Biometrika, 68, (1), 341–342.

SRIVASTAVA, S.K. and JHAJJ, H.S. (1983): A class of estimators of the population mean using multi-auxiliary information. Calcutta Statistical Association Bulletin, 32, 47–56.

SRIVENKATARAMANA, T. (1980): A dual to ratio estimator in sample surveys. Biometrika, 67, 199–204.

SRIVENKATARAMANA, T. and TRACY, D.S. (1980): An alternative to ratio method in sample surveys. Annals of Institute of Statistical Mathematics, 32, 111–120.

SRIVENKATARAMANA, T. and TRACY, D.S. (1981): Extending product method of estimation to positive correlation case in surveys. Australian Journal of Statistics.23, 1, 95–100.

TRIPATHI, T.P. (1980): A general class of estimators of population ratio. Sankhya, 42, C, 63–75.

TUTEJA, R.K. and BAHL, S. (1991): Multivariate product estimators. Calcutta Statistical Association Bulletin, 42, 109–115.

UPADHYAYA , L.N. and SINGH, H.P. (1984): Combination of regression and ratio estimate. Current Science, 53(6), 294–296.

UPADHYAYA, L.N. and SINGH, H.P. (1999): Use of transformed auxiliary variable in estimating the finite population mean. Biometrical Journal, 41, 5, 627–636.

VOS, J.W.E. (1980): Mixing of direct, ratio and product method of estimation. Statistica Neerlandica, 34(4), 209–218.

WALSH, J.E. (1970): Generalization of ratio estimate for population total. Sankhya, 32, A, 99–106.

STATISTICS IN TRANSITION-new series, July 2009
Vol. 10, No. 1, pp. 101—108

BAYESIAN ESTIMATION OF SCALE PARAMETER OF INVERSE GAUSSIAN DISTRIBUTION USING LINES LOSS FUNCTION

Himanshu Pandey and Arun Rao

ABSTRACT

In this paper we have obtained Bayes estimator of the scale parameter of Inverse Gaussian Distribution. The loss function used is lines. A number of prior distributions have been considered and Bayes estimators have been compared with corresponding estimators with squared error loss function.

1. Introduction

The Inverse Gaussian (IG) is a long-tailed positively skewed distribution. Its shape is similar to that of the normal distribution. This distribution was originally derived as the first passage of lime distribution of Brownain motion with positive drift. In reliability and life-span applications it is primarily useful when there is substantial skewness. It has been studied extensively by Tweedie (1957 a, b) and Cohen and Whitten (1988). The Probability density function (pdf) of IG distribution is given by (Chhikara and Folks, 1989)

$$f(x;\ \mu,\theta) = \left(\frac{1}{2\pi\theta}\right)^{\frac{1}{2}} x^{-\frac{3}{2}} e^{-\frac{(x-\mu)^2}{2\mu^2\theta x}} ;x > 0, \mu > 0, \theta>0 \tag{1.1}$$

where θ is the scale parameter.

Suppose $\Delta = \frac{\hat{\theta}}{\theta} - 1$. Consider the following linex loss function

$$L(\Delta) = b\,[e^{a\Delta} - a\Delta - 1], a \neq 0, b > 0 \tag{1.2}$$

where $\hat{\theta}$ is an estimate of θ (Basu and Ebrahimi, 1991). Let us denote the posterior pdf of θ by $f\ (\theta\,/\,x)$, where x denotes a random sample x = (x_1,----, x_r). Let E_π stand for posterior expectation with respect to f $(\theta/\underline{x})$. The posterior expectation of the loss function is given by equation (1.2) as

$$E_\pi[L(\Delta)] = b\left[e^a E_\pi\left(e^{a\hat{\theta}/\theta}\right) - a E_\pi\left(\frac{\hat{\theta}}{\theta} - 1\right) - 1\right] \tag{1.3}$$

The value of θ that minimizes (1.3) is denoted by $\hat{\theta}_A$ (Bayes estimator under L (Δ)) and may be obtained by solving the equation

$$\frac{d}{d\hat{\theta}} E_\pi[L(\Delta)] = 0$$

$$\Rightarrow E_\pi\left[\frac{1}{\theta}\left(e^{a\hat{\theta}/\theta}\right)\right] = e^a E_\pi\left[\frac{1}{\theta}\right] \tag{1.4}$$

if the expectations exist.

The objective of the present paper is to obtain a Bayes estimator of θ using a number of prior distributions. For the situation where we have no prior information about the parameter θ, we may use the quasi-density

$$g_1(\theta) = \frac{1}{\theta^d}\ ; \quad \theta > 0, d \geq 0 \tag{1.5}$$

here d = 0 leads to a diffuse prior and d = 1, a non-informative prior.

The most widely used prior distribution of θ is the inverted gamma with parameters $\alpha, \beta > 0$, with density function given by

$$g_2(\theta) = \begin{cases} \frac{\beta^\alpha}{\alpha}\left(\frac{1}{\theta}\right)^{\alpha+1} e^{-\beta/\theta} & ; \theta > 0 \\ 0 & ; \text{otherwise} \end{cases} \tag{1.6}$$

if it is known in advance that the probable value of θ lies over a finite range $[\alpha, \beta]$, but do not have any strong opinion about any subset of values this range. In such a case a uniform distribution over $[\alpha, \beta]$ may be appropriate.

$$g_3(\theta) = \begin{cases} \frac{1}{\beta-\alpha} & ; \ 0 < \alpha \leq \theta \leq \beta \\ 0 & ; \ \text{otherwise} \end{cases} \tag{1.7}$$

2. Bayes Estimators under $g_1(\theta)$

Let us suppose that n items are put to life test and terminate the experiment when r ($\leq n$) items have failed. If x_1,------------, x_r denote the first r observations, then the joint pdf is given by

$$f(\underline{x} : \theta) = \frac{(n)!}{(n-r)!} (Z\pi\theta)^{-r/2} \left(\prod_{i=1}^{r} X_i^{-3/2} \right) . e^{-Z/\theta} \qquad (2.1)$$

where , $Z = \frac{1}{2\mu^2} \left(\sum_{i=1}^{r} \frac{(X_i-\mu)^2}{X_i} + (n-r) \frac{(X_r-\mu)^2}{X_r} \right)$

The maximum likelihood estimator (MLE) of θ is given by

$$\hat{\theta} = \frac{2Z}{r} \qquad (2.2)$$

The posterior pdf of θ is obtained as

$$f(\theta/\underline{x}) = \frac{Z^{\left(\frac{r}{2}+d-1\right)}}{\Gamma\left(\frac{r}{2}+d-1\right)} \theta^{-\left(\frac{r}{2}+d\right)} e^{-Z/\theta} \qquad (2.3)$$

Thus, using (1.2), the Bays estimator of θ relative to L (Δ) comes out to be

$$\hat{\theta}_A = \frac{1}{a}\left[1 - e^{-a\left(\frac{r}{2}+d\right)}\right] Z \qquad (2.4)$$

Here a is positive, since we assume that overestimating θ is more costly than under estimating it. The Bayes estimator of θ under squared error loss function (SELF) is given by

$$\hat{\theta}_s = \left[\frac{Z}{\frac{r}{2}+d-2}\right] \qquad (2.5)$$

The risk function of estimators $\hat{\theta}_A$ and $\hat{\theta}_S$ relative to L (Δ) are denoted by $R_A(\hat{\theta}_A)$ and $R_A(\hat{\theta}_S)$ respectively, and are as follows

$$R_A(\hat{\theta}_A) = b\left[e^{-ad\left(\frac{r}{2}+d\right)} - \frac{r}{2}\left(1 - e^{-a\left(\frac{r}{2}+d\right)} + a - 1\right)\right] \qquad (2.6)$$

$$R_A(\hat{\theta}_S) = b\left[\left(1 - \frac{a}{\frac{r}{2}+d-2}\right)^{-\frac{r}{2}} - \left(\frac{\frac{ar}{2}}{\frac{r}{2}+d-2}\right) + a - 1\right] \tag{2.7}$$

The risk functions of the estimators $\hat{\theta}_A$ and $\hat{\theta}_S$ relative to squared error loss $\left(\theta - \hat{\theta}\right)^2$ are denoted by $R_S\left(\hat{\theta}_A\right)$ and $R_S\left(\hat{\theta}_S\right)$ respectively and we have

$$R_S\left(\hat{\theta}_A\right) = \theta^2\left[\frac{\frac{r}{2}\left(\frac{r}{2}+1\right)}{a^2}\left\{1 - e^{-a}\left(\frac{r}{2}+d\right)\right\}^2\right.$$

$$\left. - \frac{r}{a}\left\{1 - e^{-a6}\left(\frac{r}{2}+d\right) + 1\right\}\right] \tag{2.8}$$

$$R_S\left(\hat{\theta}_S\right) = \theta^2\left[\frac{\frac{r}{2}\left(\frac{r}{2}+1\right)}{\left(\frac{r}{2}+d-2\right)^2} - \frac{r}{\left(\frac{r}{2}+d-2\right)} + 1\right] \tag{2.9}$$

In general, neither of the estimators uniformly dominates the other. For example, if a = 1, r = 10, d = 1, then

$$\frac{R_S(\hat{\theta}_S)}{\theta^2} = 0.375 > 0.1719\ \frac{R_S(\hat{\theta}_A)}{\theta^2}$$

and

$$\frac{R_A(\hat{\theta}_S)}{b} = 0.3002 > 0.0789\ \frac{R_A(\hat{\theta}_A)}{b}$$

If a = 1, r = 10, d = 3, then

$$\frac{R_S(\hat{\theta}_S)}{\theta^2} = 0.1667 < 0.2392\ \frac{R_S(\hat{\theta}_A)}{\theta^2}$$

and

$$\frac{R_A(\hat{\theta}_S)}{b} = 0.0821 < 0.0998\ \frac{R_A(\hat{\theta}_A)}{b}$$

3. Bayes Estimators under g_2 (θ)

Under g_2 (θ), using (2.1), we obtain the posterior distribution as

$$f\left(\theta/\underline{x}\right) = \frac{(\beta+Z)^{\frac{r}{2}+\alpha}}{\Gamma\left(\frac{r}{2}+\alpha\right)}\, \theta^{-\left(\frac{r}{2}+\alpha+1\right)}\, e^{-(\beta+Z)/\theta} \quad ; \theta > 0 \tag{3.1}$$

Thus, using (1.2), the Bayes estimator of θ relative to L (Δ) is obtained as

$$\hat{\theta}_A = \frac{1}{\alpha}\left[1 - e^{-a\left(\frac{r}{2}+\alpha+1\right)}\right](\beta + Z) \tag{3.2}$$

The Bayes estimator of θ under SELF is given by

$$\hat{\theta}_S = \left(\frac{\beta + Z}{\frac{r}{2} + \alpha - 1}\right) \tag{3.3}$$

The risk functions of the estimators $\hat{\theta}_A$ and $\hat{\theta}_S$ relative to L $(\Delta)$are then given by

$$R_A(\hat{\theta}_A) = b\Big[\Big[\exp\left\{-a(\alpha+1)\left(\frac{r}{2}+\alpha+1\right)\right\}\Big]$$

$$\Big[\exp\Big\{\frac{\beta}{\theta}\Big[1 - \exp\Big\{-a\Big(\frac{r}{2}+\alpha+$$

$$1\Big)\Big\}\Big]\Big\}\Big] - \left[1 - \exp\left\{-a\left(\frac{r}{2}+\alpha+1\right)\right\}\right]\left(\frac{r}{2}+\frac{\beta}{\theta}\right) + a - 1\Big] \tag{3.4}$$

and,

$$R_A(\hat{\theta}_S) = b\left[\left[\exp\left\{-a\left[\frac{(1-\beta)}{\theta\left(\frac{r}{2}+\alpha+1\right)}\right]\right\}\right]\right.$$

$$\left.\left[1 - \frac{a}{\left(\frac{r}{2}+\alpha-1\right)}\right]^{-\frac{r}{2}} - \left[\frac{a\left(\frac{r}{2}+\frac{\beta}{\theta}\right)}{\left(\frac{r}{2}+\alpha-1\right)}\right] + a - 1\right] \tag{3.5}$$

The Bayes risks for the estimators $\left(\hat{\theta}_A\right)$ and $\left(\hat{\theta}_S\right)$ are denoted by $r_A\left(\hat{\theta}_A\right)$ and $r_A\left(\hat{\theta}_S\right)$ respectively, and are given by

$$r_A\left(\hat{\theta}_A\right) = b\left[a - \left(\frac{r}{2} + \alpha + 1\right)\left\{1 - \exp\left(-a\left(\frac{r}{2} + \alpha + 1\right)\right)\right\}\right] \quad (3.6)$$

and

$$r_A\left(\hat{\theta}_S\right) = b\left[e^{-a}\left[1 - \frac{a}{\left(\frac{r}{2}+\alpha-1\right)}\right]^{-\left(\frac{r}{2}+\alpha\right)} - \left[1 + \frac{a}{\left(\frac{r}{2}+\alpha-1\right)}\right]\right] \quad (3.7)$$

The risk functions and Bayes risks of $\left(\hat{\theta}_S\right)$ and $\left(\hat{\theta}_A\right)$ under SELF are similarly obtained and are given by

$$R_S\left(\hat{\theta}_A\right) = \theta^2\left[\left[\frac{1-\exp\left\{-a\left(\frac{r}{2}+\alpha+1\right)\right\}}{a}\right]^2\left[\frac{r}{2}\left(\frac{r}{2}+1\right)\right.\right.$$

$$\left. + \frac{r\beta}{\theta} + \frac{\beta^2}{\theta^2}\right] - \frac{2}{a}\left[1 - \exp\left\{-a\left(\frac{r}{2} + \right.\right.\right.$$

$$\left.\left.\left.\alpha + 1\right)\right\}\right]\left[\frac{r}{2} + \frac{\beta}{\theta}\right] + 1\Bigg] \quad (3.8)$$

$$R_S\left(\hat{\theta}_S\right) = \theta^2\left[\left[\frac{\frac{r}{2}\left(\frac{r}{2}+1\right)+\frac{r\beta}{\theta}+\frac{\beta^2}{\theta^2}}{\left(\frac{r}{2}+\alpha-1\right)^2}\right] - \frac{2\left[\frac{r}{2}+\frac{\beta}{\theta}\right]}{\left(\frac{r}{2}+\alpha-1\right)} + 1\right] \quad (3.9)$$

$$r_S\left(\hat{\theta}_A\right) = \beta^2\left[\frac{\frac{r}{2}\left(\frac{r}{2}+1\right)K^2 - rK + 1}{(\alpha-1)(\alpha-2)} + \frac{2K(rK/2-1)}{(\alpha-1)} + K^2\right] \quad (3.10)$$

where,

$$K = \frac{1}{a}\left[1 - \exp\left\{-a\left(\frac{r}{2} + \alpha + 1\right)\right\}\right]$$

and,

$$r_S(\hat{\theta}_S) = \frac{\beta^2}{(\alpha-1)(a-2)\left(\frac{r}{2}+a-1\right)} \tag{3.11}$$

4. Bayes Estimators under $g_3\,(\theta)$

Under $g_3\,(\theta)$, using (2.1), the posterior distribution is given by

$$f(\theta/\underline{x}) = \frac{Z^{\frac{r}{2}-1}\,\theta^{-\frac{r}{2}}\,e^{-z/\theta}}{Ig\left[\frac{Z}{\alpha},\frac{r}{2}-1\right]-Ig\left[\frac{Z}{\beta},\frac{r}{2}-1\right]} \qquad ; 0 < \alpha \le \theta \le \beta \tag{4.1}$$

Where $Ig(x,n) = \int_0^x e^{-t}\,t^{\,n-1}\,dt$ is the incomplete gamma function.

Using (1.2), the Bayes estimator of θ relative to L (Δ) is $\hat{\theta}_A$, where $\hat{\theta}_A$ is the solution of the following equation

$$e^{-a}\left[\frac{Ig\left(\frac{Z}{\alpha},\frac{r}{2}\right)-Ig\left(\frac{Z}{\beta},\frac{r}{2}\right)}{Ig\left(\frac{Z-a\,\hat{\theta}_A}{\alpha},\frac{r}{2}\right)-Ig\left(\frac{Z-a\,\hat{\theta}_A}{\beta},\frac{r}{2}\right)}\right] = \left[\frac{Z}{Z-a\,\hat{\theta}_A}\right]^{\frac{r}{2}} \tag{4.2}$$

The equation (4.2) can be solved numerically. The Bayes estimator of θ under SELF is given by

$$\hat{\theta}_S = \left[\frac{Ig\left(\frac{Z}{\alpha},\frac{r}{2}-2\right)-Ig\left(\frac{Z}{\beta},\frac{r}{2}-2\right)}{Ig\left(\frac{Z}{\alpha},\frac{r}{2}-1\right)-Ig\left(\frac{Z}{\beta},\frac{r}{2}-1\right)}\right]Z \tag{4.3}$$

In this case risk functions and Bayes risks cannot be obtained in a closed form.

Conclusion

From the given example in section (2), it is clear that neither of the estimators uniformly dominates the other. We therefore recommend that the estimator’s choice lies according to the value of d in the quasi density used as the prior distribution which in turn depends on the situation at hand.

Both risk functions, given by (3.4) and (3.5) under L (Δ), depend on θ and neither of the estimators uniformly dominates the other. Risk functions given by

(3.8) and (3.9) are also depend on θ and neither of these two estimators uniformly dominates the other under squared error loss.

It is clear from the equations (4.2) and (4.3) that only numerical solution exists for the estimators $\hat{\theta}_A$ and $\hat{\theta}_S$. In this case, the risk functions and Bayes risks cannot be obtained in closed forms. Thus, the comparison could only be done after obtaining the results numerically, which depends on the value of the parameter itself.

REFERENCES

BASU, A.P. and EBRAHIMI, N. (1991): Bayesian approach to life testing and reliability estimation using asymmetric loss function – Jour. Stat. Plann. Infer., 29, 21–31.

CHHIKARA, R.S. and FOLKS, J.L. (1989) : The Inverse Gaussian distribution theory, methodology and applications. Vol. 95, Marcel Dekker, Inc., New York.

COHEN, A.C. and WHITTEN, B.J. (1988) : Parameter estimation in reliability and life span models. Vol. 96, Marcel Dekker, Inc., New York.

TWEEDIE, M.C.K. (1957 a) : Statistical properties of the inverse Gaussian distribution I.Ann. Math. Stat., 28, 362–377.

TWEEDIE, M.C.K. (1957 b) : Statistical properties of the inverse Gaussian distribution II. Ann. Math. Stat., 28, 695–705.

STATISTICS IN TRANSITION-new series, July 2009
Vol. 10, No. 1, pp. 109—127

BAYES ESTIMATOR OF GENERALIZED-EXPONENTIAL PARAMETERS UNDER GENERAL ENTROPYOSS FUNCTION USING LINDLEY'S APPROXIMATION

Rahul Singh[1], S.K. Singh[2], Umesh Singh[3] and G.P.Singh[4]

ABSTRACT

In this paper, we have obtained the Bayes Estimator of scale and shape parameter of Generalized-Exponential using Lindley's approximation (L-approximation) under GENERAL ENTROPY loss functions. The proposed estimators have been compared with the corresponding MLE for their risks based on simulated samples from the Generalized-Exponential distribution.

1 Introduction

Exponential distribution is the most exploited distribution for life data analysis, but its suitability is restricted to constant hazard rate. For situations where the failure rate is monotonically increasing or decreasing, two-parameter Weibull and Gamma are the most popular distributions used for analyzing any lifetime data. Both distributions have increasing / decreasing hazard rates depending on the shape parameter. However, one of the major disadvantages of the gamma distribution is that its distribution and survival functions cannot be expressed in a closed form if the shape parameter is not an integer. Moreover, even for integer shape parameter, there are terms involving the incomplete gamma function. Thus, one needs numerical integration to obtain distribution function, survival function, or hazard function. Perhaps, this makes the gamma distribution unpopular compared to a Weibull distribution, which has a nice closed form for the hazard and survival functions. On the other hand, the Weibull distribution has

[1] Department of Adm. Reforms, Govt. of U.P, Lucknow, India.
[2] Department of Statistics and DST center for CIMS, B.H.U., Varanasi-221005, India, Email: sanjay_upc@sify.comn.
[3] Department of Statistics and DST center for CIMS, B.H.U., Varanasi-221005, India.
[4] Department of community medicine, I.M.S., B.H.U., Varanasi-221005, India.

its own disadvantages. For example, Bain and Engelhardt (1991) have pointed out that the maximum likelihood estimators of a Weibull distribution might not behave properly for all parameter ranges.

Recently a new distribution, called Generalized-Exponential distribution, has been introduced. This distribution can be used quite effectively in situations where a skewed distribution is needed. Gupta and Kundu (1999, 2002) and Raqab and Ahsanullah (2001) have investigated several properties of the two parameter generalized exponential distribution.

The two-parameter Generalized-Exponential has a distribution function of the form

$$F_{GE}\left(x \mid \alpha, \lambda\right) = \left(1 - e^{-\lambda x}\right)^{\alpha} \; ; \; \alpha \;, \; \lambda \; > \; 0 \tag{1}$$

and hazard function given by

$$h\left(x \mid \alpha, \lambda\right) = \frac{\alpha\lambda\left(1 - e^{-\lambda x}\right)^{\alpha - 1} e^{-\lambda x}}{1 - \left(1 - e^{-\lambda x}\right)^{\alpha}} \tag{2}$$

Here α is the shape parameter, and λ is the scale parameter. The two-parameter Generalized-Exponential has increasing / decreasing failure rates depending on the shape parameter. For any λ, the hazard function is increasing if $\alpha > 1$, decreasing if $\alpha < 1$, and constant if $\alpha = 1$. Gupta and Kundu (1999) observed that because of the simple structure of the distribution and survival functions, the two-parameter Generalized-Exponential can be used quite effectively in analyzing many lifetime data, particularly in place of two-parameter gamma and Weibull distributions.

The estimation of parameters of the Generalized-Exponential distribution has been attempted by Gupta and Kundu (1999), but that work was only concerned with the maximum likelihood estimator or a Bayes estimator under a symmetric loss function. It is remarkable that most of the Bayesian inference procedures have been developed with the usual squared-error loss function, which is symmetrical and associates equal importance to the losses due to overestimation and underestimation of equal magnitude. Such a restriction may be impractical in most of the situations of practical importance. For example, in the estimation of reliability and failure rate functions, an overestimation is usually much more serious than an underestimation. In this case, the use of symmetrical loss function might be inappropriate. A similar comment has been made by Basu and Ebrahimi (1991). A useful asymmetric loss known as the LINEX loss function (linear-exponential) was introduced by Varian (1975) and has been widely used by several authors, Zellner (1986), Calabria and Pulcini (1996), Soliman (2002), Singh et al. (2005), Ahmadi et al. (2005) and Singh et al. (2008). This function rises approximately exponentially on one side of zero and approximately linearly on the other side. It may also be noted here that the squared-error loss function can be obtained as a particular member of the LINEX loss function for a specific choice of the loss function parameter.

Despite the flexibility and popularity of the LINEX loss function for the location parameter estimation, it appears to be unsuitable for the scale parameter and other quantities (c.f. Basu and Ebrahimi (1991) and Parsian and Sanjari Farsipour (1993)). Keeping these points in mind, Basu and Ebrahimi (1991) defined a modified LINEX loss. A suitable alternative to the modified LINEX loss is the General ENTROPY loss proposed by Calabria and Pulcini (1994a), defined as

$$L_E\left(\bar{\alpha},\alpha\right) \propto \left(\bar{\alpha}/\alpha\right)^c - c\log\left(\bar{\alpha}/\alpha\right) - 1 \tag{3}$$

where $\bar{\alpha}$ is the estimate of parameter α.

This loss is a generalization of the ENTROPY loss used by several authors (see for example, Dey et al. (1987) and Dey and Liu (1992)) where the shape parameter c is taken equal to 1.

The general version (3) allows different shapes of the loss function. It may be noted that when c>0, a positive error causes more serious consequences than a negative error. On the other hand when c<0, a negative error causes more serious consequences than a positive error.

The Bayes estimate $\bar{\alpha}_E$ of α under General ENTROPY loss is given as

$$\bar{\alpha}_E = \left[E_\alpha\left\{\alpha^{-c}\right\}\right]^{-1/c} \tag{4}$$

provided that $E_\alpha\left\{\alpha^{-c}\right\}$ exists and is finite. It can be shown that, when c=1, the Bayes estimate (4) coincides with the Bayes estimate under the weighted squared-error loss function. Similarly, when c= -1 the Bayes estimate (4) coincides with the Bayes estimate under squared error loss function.

This paper considers the Problem of estimating the shape and scale Parameter of Generalized-Exponential distribution using General ENTROPY loss function. It is worthwhile to mention here that posterior distribution of shape and scale parameter for Generalized-Exponential distribution given in (1) involves integral expression in the denominator which can not be reduced in a nice closed form and hence the exact evaluation of posterior expectations for obtaining Bayes estimator for shape and scale parameter will not be possible. We have used Lindley Approximation to obtain Bayes estimators under symmetric and asymmetric loss function.

In the last section, the proposed estimators have been compared with those under quadratic loss function in term of their risk. The comparison is based on

Monte-Carlo study of 2000 simulated sample from Generalized-Exponential distribution.

2. Estimation of parameters

Suppose $x_1,x_2,\ldots,x_n$ is a random sample of size n from the distribution function defined in (2). The likelihood function of λ and α for the samples $x_1,x_2,\ldots,x_n$ is

$$L(\lambda,\alpha \mid x)=\alpha^n\lambda^n\left[\prod_{x=1}^{n}\left(1-e^{-\lambda x_i}\right)\right]^{\alpha-1}\exp\left[-\lambda\sum_{i=1}^{n}x_i\right]:x_i\geq 0 \quad (5)$$

2.1. Maximum Likelihood Estimators of Generalized-Exponential Distributions

The maximum likelihood estimate of parameters of the Generalized-Exponential distribution is obtained by differentiating the log of the likelihood and equating to zero. The two normal equations thus obtained are given below:

$$\frac{n}{\lambda}-\left(\frac{n}{\sum_{i=1}^{n}\log\left(1-e^{-\lambda x_i}\right)}+1\right)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{\left(1-e^{-\lambda x_i}\right)}-\sum_{i=1}^{n}x_i=0 \qquad (6)$$

and

$$\hat{\alpha}=\left(\frac{n}{\sum_{i=1}^{n}\log\left(1-e^{-\lambda x_i}\right)}+1\right) \qquad (7)$$

But these normal equations are not solvable. Therefore, the MLE does not exist in a nice closed form. However, the maximum likelihood estimator of two-parameter Generalized-Exponential distribution can be obtained by iterative procedures. We propose here to use bisection method for solving the above-mentioned normal equations.

2.2. Bayes Estimator of Generalized-Exponential Distributions

For Bayesian estimation, we need prior distribution for the parameters α and λ. It may be noted here that when the shape parameter is equal to one, the generalized exponential distribution reduces to exponential distribution. Hence, gamma prior may be taken as the prior distribution for the scale parameter of the Generalized-Exponential distribution. It may be noted here that under the above-mentioned situation, the gamma prior is a conjugate prior. On the other hand, if both the parameters are unknown, a joint conjugate prior for the parameters does not exist. In such a situation, there are a number of ways to choose the priors. We propose the use of piecewise independent priors for both the parameters, namely a non-informative prior for the shape parameter and a natural conjugate prior for the scale parameter (under the assumption that shape parameter is known). Thus the proposed priors for parameters α and λ may be taken as

$$g_1(\alpha) \propto \frac{1}{\alpha}, \quad 0 \le \alpha < \infty, \tag{8}$$

and

$$g_2(\lambda) \propto \frac{m^c \lambda^{c-1} \exp(-m\lambda)}{\Gamma(c)}, \quad \lambda \ge 0; m, c > 0 \tag{9}$$

respectively, to give the joint prior distribution for λ and α as:

$$g(\alpha,\lambda) = \frac{m^c \lambda^{c-1} \exp(-m\lambda)}{\alpha\Gamma(c)}, \lambda \ge 0; m, c > 0 \quad 0 \le \alpha < \infty \tag{10}$$

Substituting $L(\alpha,\lambda \mid x)$ and $g(\alpha,\lambda)$ from (5) and (10) respectively in the following equation

$$P(\alpha,\lambda \mid x) = \frac{L(\alpha,\lambda \mid x).g(\alpha,\lambda)}{\int_0^\infty \int_0^\infty L(\alpha,\lambda \mid x).g(\alpha,\lambda)\, d\lambda\, d\alpha},$$

we get the joint posterior $P(\alpha,\lambda \mid x)$ as

$$P(\alpha,\lambda \mid x) = \frac{\alpha^{n-1}\lambda^{n+c-1}\left[\prod_{x=1}^{n}\left(1-e^{-\lambda xi}\right)\right]^{\alpha-1} \exp\left[-\lambda\left(m+\sum_{i=1}^{n} X_i\right)\right]}{\int_0^\infty \int_0^\infty \alpha^{n-1}\lambda^{n+c-1}\left[\prod_{x=1}^{n}\left(1-e^{-\lambda xi}\right)\right]^{\alpha-1} \exp\left[-\lambda\left(m+\sum_{i=1}^{n} X_i\right)\right] d\lambda d\alpha} \tag{11}$$

Note that the posterior distribution of (α,λ) takes a ratio form, that involves integration in the denominator, which cannot be reduced in nice closed form. Hence, the evaluation of the posterior expectation for obtaining the Bayes estimator of α and λ will be tedious and it will be ratio of unsolvable integrals. Among the various methods suggested to approximate such ratio of integrals, perhaps the simplest one is Lindley's approximation method, which approximates the ratio of the integrals as a whole and produces a single numerical result. Thus, we propose the use of Lindley's (1980) approximation for obtaining the Bayes estimator of α and λ. Many authors have used this approximation for obtaining the Bayes estimators for some lifetime distributions; see among others, Howlader and Hossain (2002) and Jaheen (2005), Singh et al. (2008).

If n is sufficiently large, according to Lindley (1980), any ratio of the integral of the form

$$I(x)=E\left[u(\lambda,\alpha\mid x)\right]=\frac{\int_{(\lambda,\alpha)} u(\lambda,\alpha)e^{L(\lambda,\alpha)+G(\lambda,\alpha)}d(\lambda,\alpha)}{\int_{(\lambda,\alpha)} e^{L(\lambda,\alpha)+G(\lambda,\alpha)}d(\lambda,\alpha)}, \tag{12}$$

Where $u(\lambda,\alpha)$ = function of λ and α only ;

$L(\lambda,\alpha)$= log of likelihood;

$G(\lambda,\alpha)$= log of joint prior of λ and α .

can be evaluated as

$$I(x)=u(\hat{\lambda},\hat{\alpha})+\frac{1}{2}\Big[(\hat{u}_{\lambda\lambda}+2\hat{u}_{\lambda}\hat{p}_{\lambda})\hat{\sigma}_{\lambda\lambda}+(\hat{u}_{\alpha\lambda}+2\hat{u}_{\alpha}\hat{p}_{\lambda})\hat{\sigma}_{\alpha\lambda}$$

$$+(\hat{u}_{\lambda\alpha}+2\hat{u}_{\lambda}\hat{p}_{\alpha})\hat{\sigma}_{\lambda\alpha}+(\hat{u}_{\alpha\alpha}+2\hat{u}_{\alpha}\hat{p}_{\alpha})\hat{\sigma}_{\alpha\alpha}\Big]$$

$$+\frac{1}{2}\begin{bmatrix}(\hat{u}_{\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{u}_{\alpha}\hat{\sigma}_{\lambda\alpha})(\hat{L}_{\lambda\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\lambda\alpha\lambda}\hat{\sigma}_{\lambda\alpha}+\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\alpha\lambda}+\hat{L}_{\alpha\alpha\lambda}\hat{\sigma}_{\alpha\alpha})+\\(\hat{u}_{\lambda}\hat{\sigma}_{\alpha\lambda}+\hat{u}_{\alpha}\hat{\sigma}_{\alpha\alpha})(\hat{L}_{\alpha\lambda\lambda}\hat{\sigma}_{\lambda\lambda}+\hat{L}_{\lambda\alpha\alpha}\hat{\sigma}_{\lambda\alpha}+\hat{L}_{\alpha\lambda\alpha}\hat{\sigma}_{\alpha\lambda}+\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha})\end{bmatrix} \tag{13}$$

where $\hat{\lambda}$ = M.L.E. of λ ;

$\hat{\alpha}$ = M.L.E. of α.

$$\hat{u}_{\lambda}=\frac{du(\hat{\lambda},\hat{\alpha})}{d\hat{\lambda}};\hat{u}_{\alpha}=\frac{du(\hat{\lambda},\hat{\alpha})}{d\hat{\alpha}};\hat{u}_{\lambda\alpha}=\frac{d^{2}u(\hat{\lambda},\hat{\alpha})}{d\hat{\lambda}d\hat{\alpha}};$$

$$\hat{u}_{\alpha\lambda} = \frac{d^2u\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\alpha}\,d\hat{\lambda}}\hat{u}_{\lambda\lambda} = \frac{d^2u\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\lambda}^2};\hat{u}_{\alpha\alpha} = \frac{d^2u\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\alpha}^2}$$

$$\hat{L}_{\lambda\lambda\alpha} = \frac{d^3L\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\lambda}.d\hat{\lambda}.d\hat{\alpha}};\hat{L}_{\lambda\lambda\lambda} = \frac{d^3L\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\lambda}.d\hat{\lambda}.d\hat{\lambda}};\hat{L}_{\lambda\alpha\lambda} = \frac{d^3L\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\lambda}.d\hat{\alpha}.d\hat{\lambda}};$$

$$\hat{L}_{\alpha\alpha\lambda} = \frac{d^3L\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\alpha}.d\hat{\alpha}.d\hat{\lambda}};\hat{L}_{\alpha\lambda\lambda} = \frac{d^3L\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\alpha}.d\hat{\lambda}.d\hat{\lambda}};$$

$$\hat{L}_{\lambda\lambda\alpha} = \frac{d^3L\left(\lambda,\alpha\right)}{d\hat{\lambda}.d\hat{\lambda}.d\hat{\alpha}};\hat{L}_{\alpha\alpha\alpha} = \frac{d^3L\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\alpha}.d\hat{\alpha}.d\hat{\alpha}};\hat{L}_{\alpha\lambda\alpha} = \frac{d^3L\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\alpha}\,d\hat{\lambda}.d\hat{\alpha}};$$

$$;\hat{P}_{\alpha} = \frac{dG\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\alpha}};\hat{P}_{\lambda} = \frac{dG\left(\hat{\lambda},\hat{\alpha}\right)}{d\hat{\lambda}};$$

2.2.1 Bayes Estimator of λ

The Bayes estimator of λ under General Entropy loss function is given as

$$\hat{\lambda}_{GE} = \left[E_{\lambda}\left(\lambda^{-c_1}\right)\right]^{-\frac{1}{c_1}} \tag{14}$$

where E_{λ} stands for posterior expectations and is defined below

$$E_{\lambda}\left(\lambda^{-c_1}\right) = \int_{(\lambda,\alpha)} \lambda^{-c_1} p\left(\lambda,\alpha\right) d\left(\lambda,\alpha\right)$$

$$= \frac{\int_{(\lambda,\alpha)} u\left(\lambda,\alpha\right) e^{L(\lambda,\alpha)+G(\lambda,\alpha)} d\left(\lambda,\alpha\right)}{\int_{(\lambda,\alpha)} e^{L(\lambda,\alpha)+G(\lambda,\alpha)} d\left(\lambda,\alpha\right)}$$

Here,

$$u\left(\hat{\lambda},\hat{\alpha}\right)=\lambda^{-c_1}$$

$$L\left(\hat{\lambda},\hat{\alpha}\right) = n\log\alpha + n\log\lambda + (\alpha-1)\left[\sum_{i=1}^{n}\log\left(1-e^{-\lambda x_i}\right)\right] - \lambda\sum_{i=1}^{n}x_i$$

$$P\left(\hat{\lambda},\hat{\alpha}\right) = c\log m - \log\alpha - \log\Gamma c + (c-1)\log\lambda - m\lambda$$

Evaluating u-terms, L-terms, and p-terms mentioned above at point ($\hat{\lambda},\hat{\alpha}$) and using (13) we get

$$E_\lambda\left[\lambda^{-c_1}\right]=\lambda^{-c_1}+\frac{1}{2}\left[\left(\frac{c_1(c_1+1)\lambda^{-(c_1+2)}-2c_1\lambda^{-(c_1+1)}\left(\dfrac{c-1}{\lambda}-m\right)}{\left(\dfrac{n}{\lambda^2}+(\alpha-1)\left[\displaystyle\sum_{i=1}^{n}\frac{x_i{}^2e^{-\lambda x}}{\left(1-e^{-\lambda x_i}\right)^2}\right]\right)}\right)\right.$$

$$\left.-\left(\frac{c_1\lambda^{-(c_1+1)}\left(\dfrac{2n}{\lambda^3}+(\alpha-1)\displaystyle\sum_{i=1}^{n}\frac{x_i{}^3e^{-\lambda x_i}\left(1+e^{-\lambda x_i}\right)}{\left(1-e^{-\lambda x_i}\right)^3}\right)}{\left(\dfrac{n}{\lambda^2}+(\alpha-1)\left[\displaystyle\sum_{i=1}^{n}\frac{x_i{}^2e^{-\lambda x}}{\left(1-e^{-\lambda x_i}\right)^2}\right]\right)^2}\right)\right]$$

Thus, Bayes estimator of λ under General Entropy loss function

$$\hat{\lambda}_{GE}=\left[E_\lambda\left(\lambda^{-c_1}\right)\right]^{-\frac{1}{c_1}}$$

$$=\lambda^{-c_1}+\frac{1}{2}\left[\left(\frac{c_1(c_1+1)\lambda^{-(c_1+2)}-2c_1\lambda^{-(c_1+1)}\left(\dfrac{c-1}{\lambda}-m\right)}{\left(\dfrac{n}{\lambda^2}+(\alpha-1)\left[\displaystyle\sum_{i=1}^{n}\frac{x_i{}^2e^{-\lambda x}}{\left(1-e^{-\lambda x_i}\right)^2}\right]\right)}\right)\right.$$

$$-\left(\frac{c_1\lambda^{-(c_1+1)}\left(\frac{2n}{\lambda^3}+(\alpha-1)\sum_{i=1}^{n}\frac{x_i^{\,3}e^{-\lambda x_i}\left(1+e^{-\lambda x_i}\right)}{\left(1-e^{-\lambda x_i}\right)^3}\right)}{\left(\frac{n}{\lambda^2}+(\alpha-1)\left[\sum_{i=1}^{n}\frac{x_i^{\,2}e^{-\lambda x}}{\left(1-e^{-\lambda x_i}\right)^2}\right]\right)^2}\right)\Bigg]\Bigg]$$

It may be noted that as n$\to\infty$, $\hat{\lambda}_{GE}\to\hat{\lambda}$ i.e. when sample size n is sufficiently large, the Bayes estimator of Generalized-Exponential parameter λ under General Entropy loss function tends to maximum likelihood estimator of λ.

2.2.2 Bayes Estimator of α

The Bayes estimator of α under General Entropy loss function is

$$\hat{\alpha}_{GE}=\left[E_{\alpha}\left(\alpha^{-c_1}\right)\right]^{-\frac{1}{c_1}}$$

where E_{α} stands for posterior expectations and is defined below:

$$E_{\alpha}\left(\alpha^{-c_1}\right)=\int_{(\lambda,\alpha)}\alpha^{-c_1}p(\lambda,\alpha)d(\lambda,\alpha)$$

$$=\frac{\int_{\hat{\lambda},\hat{\alpha}}u\left(\hat{\lambda},\hat{\alpha}\right)e^{L\left(\hat{\lambda},\hat{\alpha}\right)+p\left(\hat{\lambda},\hat{\alpha}\right)}d(\lambda,\alpha)}{\int_{\hat{\lambda},\hat{\alpha}}e^{L\left(\hat{\lambda},\hat{\alpha}\right)+p\left(\hat{\lambda},\hat{\alpha}\right)}d(\lambda,\alpha)}$$

Here,

$$u\left(\hat{\lambda},\hat{\alpha}\right)=\alpha^{-c_1},$$

$$L\left(\hat{\lambda},\hat{\alpha}\right)\ =n\log\alpha+n\log\lambda+(\alpha-1)\left[\sum_{i=1}^{n}\log\left(1-e^{-\lambda x_i}\right)\right]-\lambda\sum_{i=1}^{n}x_i$$

$$P\left(\hat{\lambda},\hat{\alpha}\right)\ =\ c\log m-\log\alpha-\log\Gamma c+(c-1)\log\lambda-m\lambda$$

Evaluating u-terms, L-terms, and p-terms using (13) we get

$$E_\alpha\left[\alpha^{-c_1}\right] = \alpha^{-c_1}\left\langle 1+\frac{1}{2}c_1\frac{\alpha^2}{n}\left[\left\{(c_1+3)\alpha^{-2}\right\}\right.\right.$$

$$\left.\left.-\alpha^{-1}\left(\frac{2}{\alpha}-\frac{\sum_{i=1}^{n}\frac{x_i^2e^{-\lambda x}}{\left(1-e^{-\lambda x_i}\right)^2}}{\frac{n}{\lambda^2}+(\alpha-1)\left[\sum_{i=1}^{n}\frac{x_i^2e^{-\lambda x}}{\left(1-e^{-\lambda x_i}\right)^2}\right]}\right)\right]\right\rangle$$

Thus, the Bayes estimator of α under General Entropy loss function is given by

$$\hat{\alpha}_{GE} = \left[E_\alpha\left(\alpha^{-c_1}\right)\right]^{-\frac{1}{c_1}}$$

$$\hat{\hat{\alpha}}_{GE} = \left[\hat{\alpha}^{-c_1}\left\langle 1+\frac{1}{2}c_1\frac{\hat{\alpha}^2}{n}\left[\left\{(c_1+3)\hat{\alpha}^{-2}\right\}\right.\right.\right.$$

$$\left.\left.\left.-\hat{\alpha}^{-1}\left(\frac{2}{\hat{\alpha}}-\frac{\sum_{i=1}^{n}\frac{x_i^2e^{-\hat{\lambda}x_i}}{\left(1-e^{-\hat{\lambda}x_i}\right)^2}}{\frac{n}{\hat{\lambda}^2}+(\hat{\alpha}-1)\left[\sum_{i=1}^{n}\frac{x_i^2e^{-\hat{\lambda}x_i}}{\left(1-e^{-\hat{\lambda}x_i}\right)^2}\right]}\right)\right]\right\rangle\right]^{-\frac{1}{c_1}}$$

It may be noted that as $n \to \infty$, $\hat{\alpha}_{GE} \to \hat{\alpha}$ i.e.

when sample size n is sufficiently large then Bayes estimator of Generalized-Exponential parameter α under General Entropy loss function tends to maximum likelihood estimator of α .

3. Comparison

In the previous section we have seen that the Lindley's approximation provides the Bayes estimators in a nice closed form as function of maximum likelihood estimators. But, the maximum likelihood estimators are not expressible in closed form. Therefore, analytical expressions for the risk of the proposed estimator as well as maximum likelihood estimator cannot be obtained. Under such a situation comparison of the proposed estimator with maximum likelihood estimator based on their risk can only be studied using simulation technique. It may be noted that the risk of the estimators will be function of sample size, population parameters, parameters of the prior distribution (hyper-parameters) and loss function parameters. We have obtained the simulated risks for sample size n = 10(10)40. The different values of parameters of the distribution considered by us are loss parameter c_1 =0.5(0.5)2.0; scale parameter of model λ = 1.5(0.5)3.0; shape parameter of model α = 1.5(0.5)3.0; and prior parameters m = 1(1)5 and c = 1(1)5. For all the combination of the above-mentioned values of sample size and other parameters, risks of the estimators have been estimated on the basis of 2000 simulated samples. After an extensive study of the results, thus obtained, conclusions were drawn regarding the behaviour of the estimator which is summarized below. However, only a few results are shown in the form of graphs due to paucity of space.

While studying the effect of variation in the value of scale parameter (m) and shape parameter (c) of prior distribution, it is noted that risk of the proposed Bayes estimator of λ increases as (c) increases and decreases as (m) increases, although the variation is very little. Needless to mention that risks of maximum likelihood estimator of λ and α and proposed Bayes estimator of α are constant as there is no effect of (c) and (m) on these. Keeping this in mind, only for c=1 and m=1, the graphs of risks are included in this paper. It may be noted here that the trend is almost same as discussed above for positive and negative value of c_1 . It may also be worthwhile to mention that the risks of proposed Bayes estimators are always found to be less then those of maximum likelihood estimator.

Let us consider the effect of sample size n on the risk of the estimators, when loss parameter, prior parameter and model parameter are fixed at any particular value. For positive value of c_1 i.e. when overestimation is more serious than underestimation. It was noted from simulated risks that as sample size increases, the risks of estimator of λ as well as the risks of the estimator of α decreases, which is an obvious result. Further, it may be seen that in most of the cases, the risk of Bayes estimator of α is less than that of the risk of the maximum likelihood estimator. It may be also noted that as sample size increases the difference between the risk of Bayes estimator and maximum likelihood estimator decreases. For negative value of c_1, i.e. when underestimation is more serious than overestimation, then also, risks of estimator of λ and α decrease with increase in sample size. The risks of proposed Bayes estimator of λ and α are always less than the risk of maximum likelihood estimator of λ and α. (See figure 1, 2, 3, 4)

Now we observe the effect of **scale parameter** λ on the risks of estimator of λ keeping values of other parameters fixed for positive value of c_1. It may be noted that as λ increases, risks of estimator of λ first decreases and then increases. The risk of the maximum likelihood estimator first decreases, then increases and again increases for further increases in λ but variation is less in magnitude than that of Bayes estimator. For negative value of c_1, the risks of estimator of λ first decreases and then increases for increase in the value of λ. The maximum likelihood estimator has similar trend. It is also noticed that risks of proposed Bayes estimator of λ is always less than that of Maximum Likelihood estimator of λ which shows that proposed Bayes estimator performs well in whole of the considered values of the parameters (see figure 5, 6).

Studying the variation of **shape parameter** α on the risks of the estimator of α, it is noted that for positive value of c_1, as α increases the risk of the proposed Bayes estimator as well as maximum likelihood estimator increases. However, the risk of Bayes estimator is always less then that of maximum likelihood estimator. For negative value of c_1, i.e. when underestimation is more serious than overestimation, the risks of Bayes estimator as well as maximum likelihood estimator of α increases as value of α increases, then decreases and again increases for further increases in the value α. However, the magnitude of risk of Bayes estimator is almost equal or slightly less then that of maximum likelihood estimator (see figure 7, 8).

From the results, it is observed that for small value of c_1 the risks of Bayes estimator is increasing slightly less than the risk of the maximum likelihood estimator. However, as the magnitude of loss parameter c_1 increases, risks of the estimators of λ and α increases. It is also noticed that increases in the risks of maximum likelihood estimator are more significant than those of Bayes estimator, giving a greater gain due to use of Bayes estimator for large value of $|c_1|$. It may also be noted here that difference in risk of Bayes and maximum likelihood estimator is more for positive value of c_1 then that of negative values of c_1 (see figure 9, 10).

4. Conclusion

The behaviour of the proposed Bayes estimator under General Entropy loss in comparison to maximum likelihood estimator has been studied in above section. It is noted that for positive c_1, i.e. when overestimation is more serious than underestimation, proposed Bayes estimator of λ performs better than maximum likelihood estimator of λ for large portion of parametric space and proposed Bayes estimator of α performs better than maximum likelihood estimator of α for whole parametric space. For negative c_1, i.e. when underestimation is more serious than overestimation, proposed Bayes estimator of λ and α performs better than maximum likelihood estimator of λ and α respectively for whole parametric space. Thus, the proposed Bayes estimator may be recommended for its use.

Figure 1. Risk of estimator of λ when c_1 is positive as function of sample size N

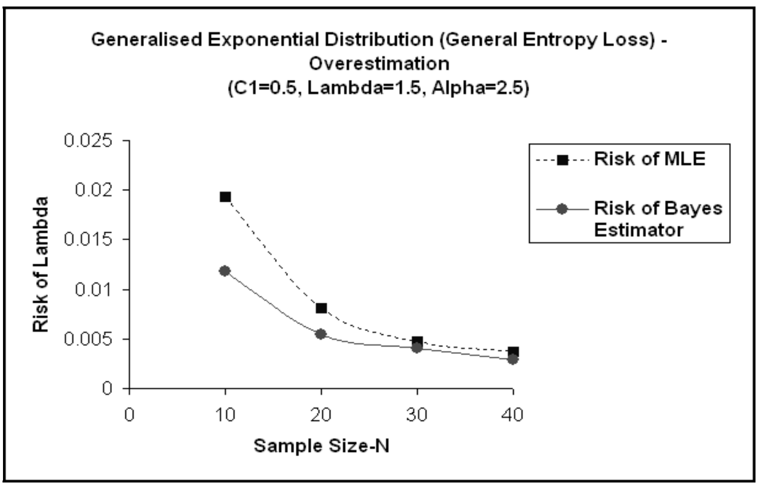


Figure 2. Risk of estimator of λ when c_1 is negative as function of sample size N

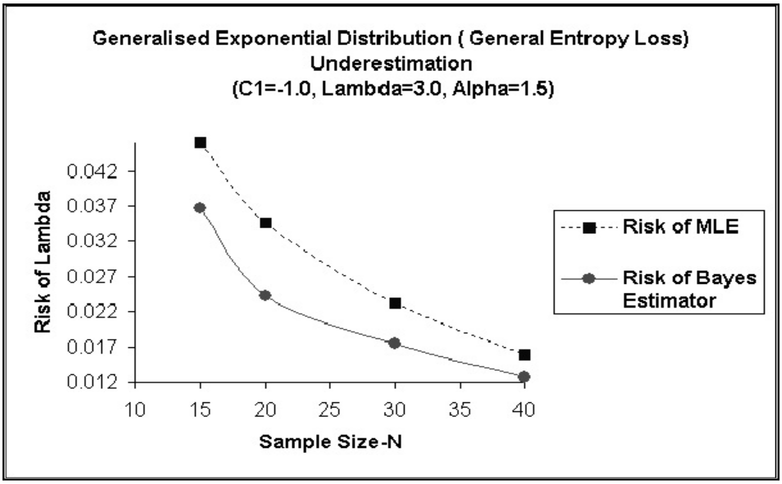


Figure 3. Risk of estimator of α when c_1 is positive as function of sample size of N

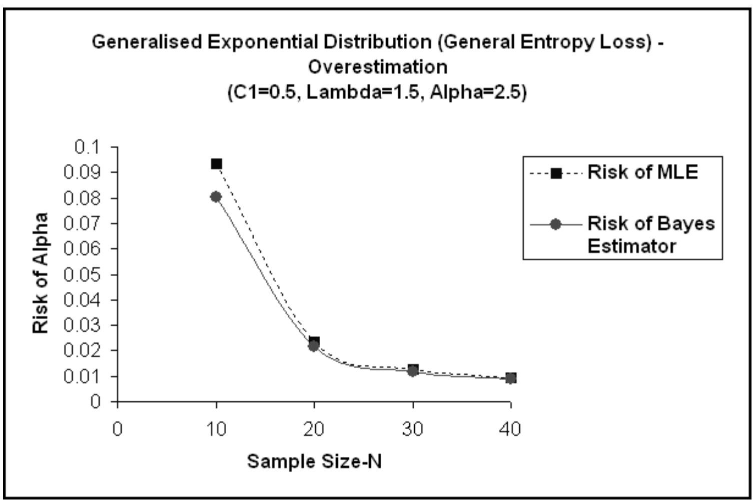


Figure 4. Risk of estimator of α when c_1 negative as function of sample size of N

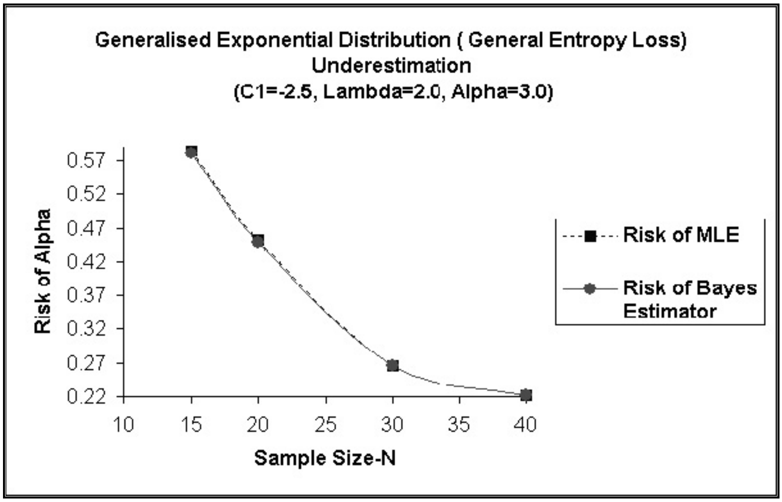


Figure 5. Risk of estimator of λ when c_1 is positive as function λ

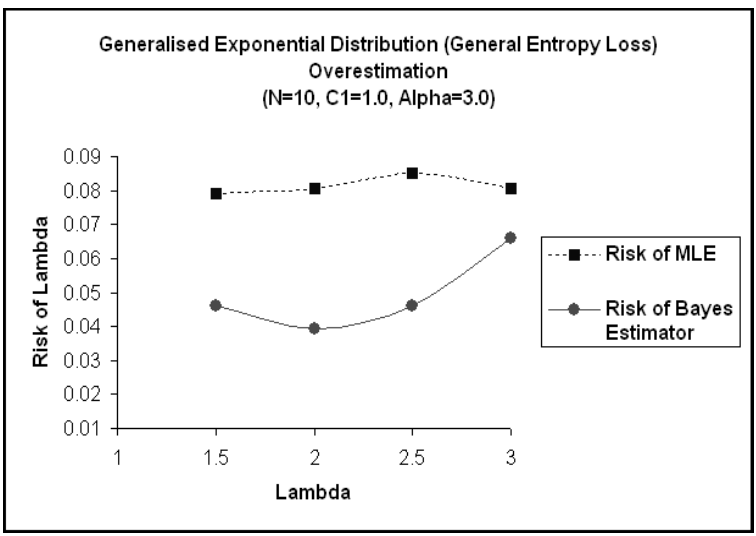


Figure 6. Risk of estimator of λ when c_1 negative as function λ

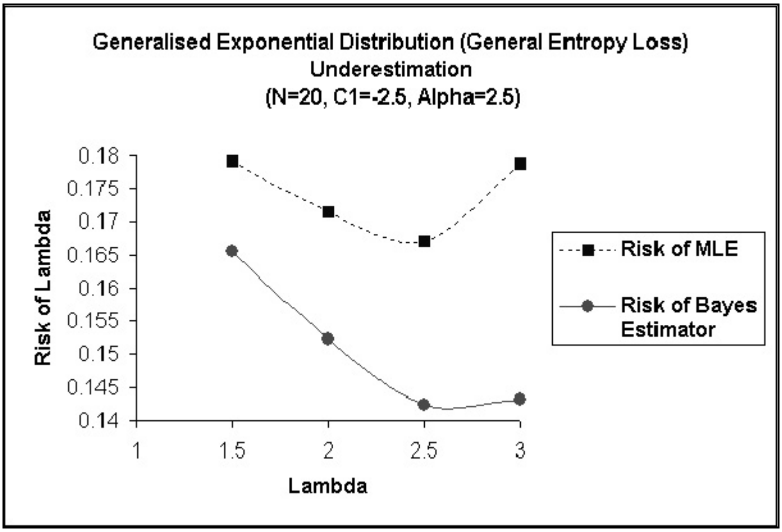


Figure 7. Risk of estimator of α when c_1 is positive as function α

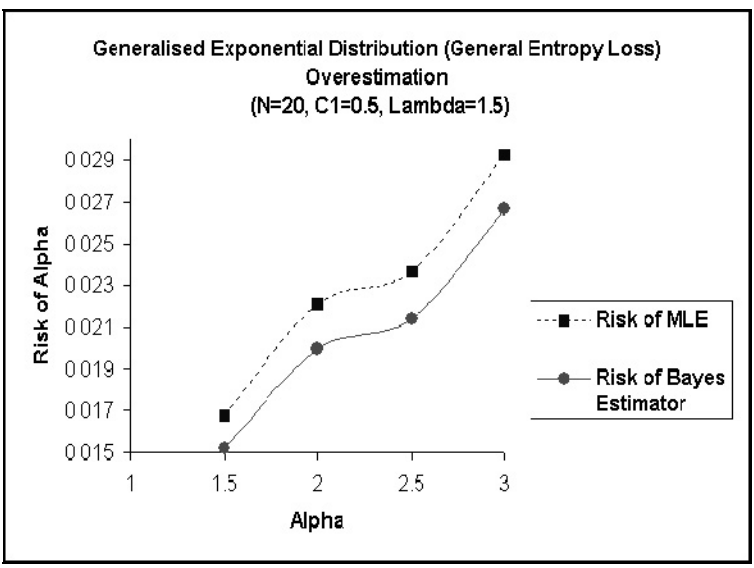


Figure 8. Risk of estimator of α when c_1 negative as function α

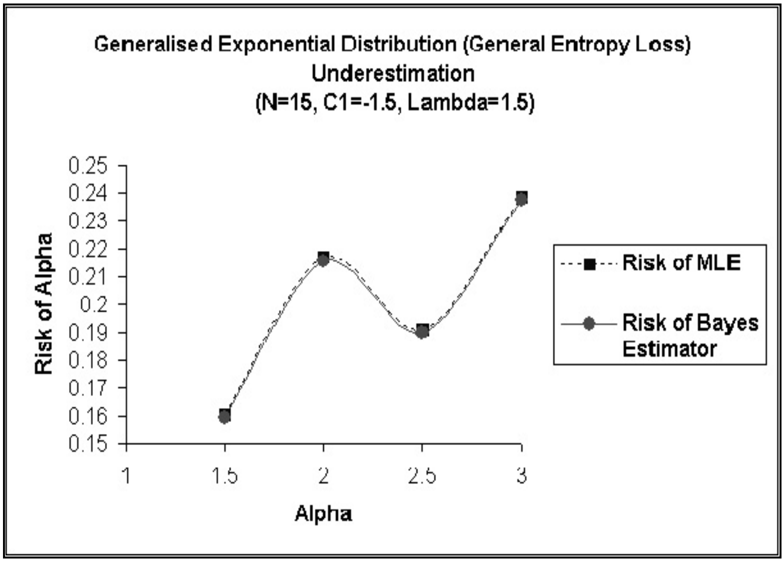


Figure 9. Risk of estimator of λ as function c_1

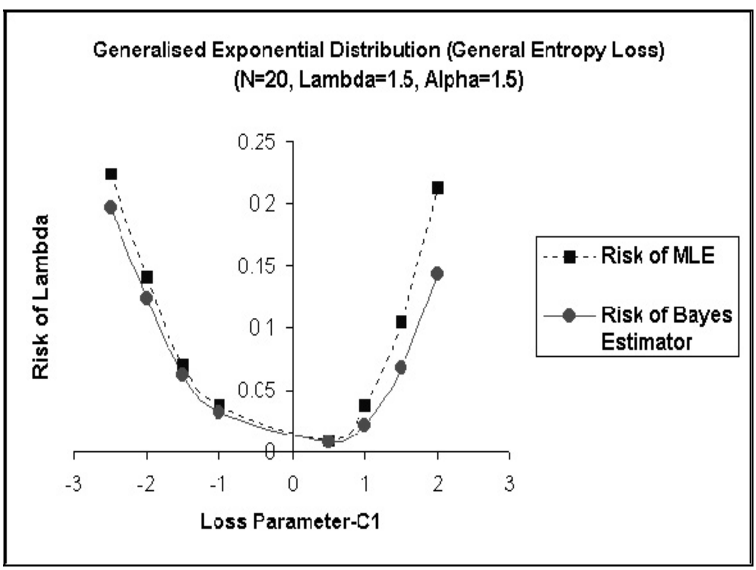
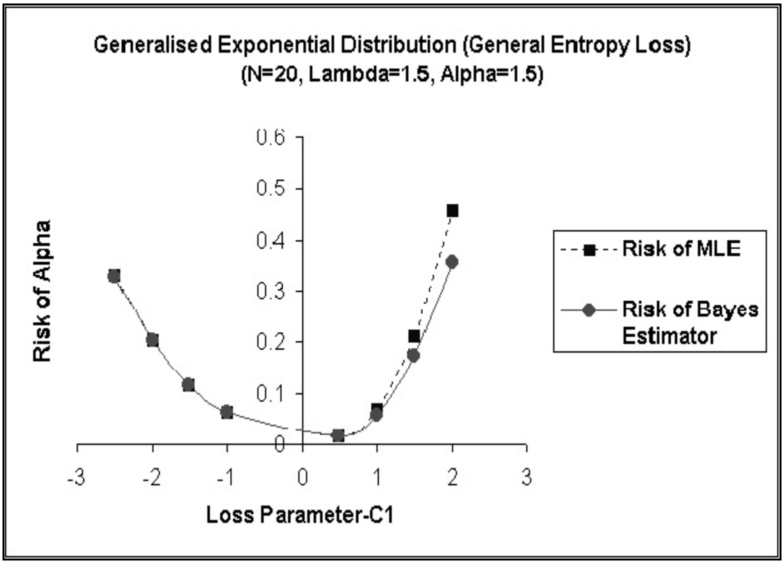


Figure 10. Risk of estimator of α as function c_1



REFERENCES

AHMADI, J., DOOSTPARAST, M., & PARSIAN, A. (2005) Estimation and prediction in a two parameter exponential distribution based on k-record values under LINEX loss function. *Commun. Statist. Theor. Meth*. 34:795–805.

BAIN, L.J. & ENGELHARDT, M. (1991) *Statistical Analysis of Reliability and Life Testing Models – Theory and Methods*. New York: Marcel Dekker, Inc.

BASU, A. P. & EBRAHIMI, N. (1991) Bayesian approach to life testing and reliability estimation using asymmetric loss-function. *J. Statist. Plann. Infer.* 29:21–31.

CALABRIA, R. and PULCINI G. (1994 a); "An engineering approach to Bayes estimation For the Weibull distribution". Microelectron Relib, 34, 789–802.

CALABRIA, R. & PULCINI, G. (1996) Point estimation under asymmetric loss functions for left truncated exponential samples. *Commun. Statist. Theor. Meth*. 25(3):585–600.

DEY, D.K., GHOSH, M. and SRINIVASAN,C. (1987)."Simultaneous estimation of Parameters under entropy loss", J. Statist. Plann. Inference, 347–363.

DEY, D.K. and Pei-San LIAO LIU (1992). " On comparison of estimators in a generalized life model", Microelectron. Reliab., 32, 207–221.

GUPTA, R.D. & KUNDU, D. (1999) Generalised-Exponential Distribution, *Australia and New Zealand Journal of Statistics*, 41,173–188.

GUPTA, R.D. & KUNDU, D. (2002) Generalised-Exponential Distribution, *Journal of Applied Statistical Society.*

JAHEEN, Z. F. (2005) On record statistics from a mixture of two exponential distributions. *J. Statist. Computat. Simul.* 75(1):1–11.

LINDLEY, D.V. (1980) Approximate Bayes methods. *Bayesian Statistics*, Valency.

Raqab, M.Z. & Ahsanullah, M (2001) Estimation of location and scale parameter of Generalised-Exponential Distribution based on Order Statistics. *Journal of Statistical Computation and Simulati*on, 69, 2, 109–124.

SINGH, U., GUPTA, P. K., & UPADHYAY, S. K. (2005) Estimation of parameters for exponentiated-Weibull family under Type-II censoring scheme. O*mputat. Statist. DataAnal.* 48(3):509–523.

PARSIAN,A. and SANJARI FARSIPOUR,N(1993). "On the admissibility and inadmissibility of estimators of scale parameters using an asymmetric loss function", Commun. Statist. Theory Meth, 22,2877–2901.

P.K. SINGH, S.K. SINGH and U. SINGH (2008) Bayes Estimator of Inverse Gaussian Parameters Under General Entropy Loss Function Using Lindley's Approximation. Appearing in *Vol. 37, Issue 4, Communications in statistics: Simulation and Computation*

SOLIMAN, A. A. (2002) Reliability estimation in a generalized life-model with application to the Burr-XII. *IEEE Trans. Reliabil.* 51: 337–343.

VARIAN, H. R. (1975) A Bayesian approach to real state assessment. In: Stephen, E. F. & Zellner, A. (Eds.) *Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage*, pp. 195–208. Amsterdam: North-Holland.

ZELLNER, A. (1986) A Bayesian Estimation and Prediction using Asymmetric Loss function. *JASA*, 81, 446–451.

STATISTICS IN TRANSITION-new series, July 2009
Vol. 10, No. 1, pp. 129—151

HIGH-VOLUME RETURN PREMIUM: AN EVENT STUDY APPROACH

Henryk Gurgul[1], Tomasz Wójtowicz

ABSTRACT

The dynamic relationships between extreme trading volume and subsequent stock returns on the Warsaw Stock Exchange, the London Stock Exchange, the Frankfurt Stock Exchange and the Vienna Stock Exchange are compared using event study methodology. The dynamic relationship between extreme trading volume and mean abnormal returns on days following an event depends on the stock exchange. This relation is mostly significant and positive in the case of the WSE, the LSE and the VSE, and depends on the nature and size of the stock exchange. The high-volume-return premium is more pronounced for small size stocks with lower liquidity levels.

Key words: extreme volume; high-volume return premium; investment strategy.

1. Introduction

Statistical investigations of stock markets concentrate primarily on stock prices and their behaviour over time. Conditional upon the available set of information about a company, its stock price reflects investors' expectations concerning the future performance of the firm. The arrival of new information causes investors to adapt their expectations and is the main source of price changes. However, since investors are heterogeneous in their interpretations of new information, prices may remain unchanged even though significant new information is revealed to the market. This will be the case if some investors interpret the news differently i.e. some of them evaluate it as good whereas others find it to be bad. Another situation in which relevant new information may leave stock prices unchanged can occur when investors interpret the information quite similarly but start with diverse prior expectations. Hence, changes in stock prices reflect an aggregation or averaging of investors' adapted beliefs.

[1] For correspondence: Henryk Gargul, Faculty of Management, University of Science and Technology, Al. Mickiewicza 30, 30-059 Kraków, Poland; E–mail: h.gurgul@neostrada.pl.

On the other hand, it is clear that stock prices may only change if there is positive trading volume. The most important question arising from this is whether volume data are simply a descriptive parameter of the trading process or whether they may contain unique information that can be exploited for modelling stock returns or return volatilities. As with prices, trading volume and volume changes mainly reflect the available set of relevant information on the market. Unlike stock prices, however, a revision in investors' expectations always leads to an increase in trading volume which therefore reflects the sum of investors' reactions to news. This summation process, which leads to trading volume, preserves differences existing between investors' reactions to the arrival of new information. These differences may get lost in the averaging process that fixes prices. Studying the joint dynamics of stock prices and trading volume therefore makes for a better understanding of the dynamic properties of stock prices and trading volume.

The main goal of this paper is to examine the impact of extreme trading volume values on returns of companies listed on the Warsaw Stock Exchange (WSE) – an emerging market, the Vienna Stock Exchange (VSE) – its local rival in Central Europe, nearly of the same size as WSE, the London Stock Exchange (LSE) – the largest stock market in Europe, and on the Frankfurt Stock Exchange (FSE) – the stock market of the largest economy within The European Union. We are interested in the explanatory power of trading volume in forecasting the direction of price movements. The efficient market hypothesis assumes that trading volume should not have any impact on future price movement. This hypothesis is put in question in the light of empirical investigations (e.g. Gervais et al. (2001)), so it may be proposed that individual stocks whose trading activity is unusually large (small) over a trading day, as measured by trading volume on this day, tend to exhibit large (small) returns over the next days. Another goal is to prove whether this hypothesis is true for emerging markets as represented by the WSE. We will show both similarities and differences between the performance of this emerging market and well developed capital markets under this hypothesis. In particular we will prove the impact of the size of the companies under study on the validity of the high-volume return premium hypothesis. We also will check the influence of bull (bear) markets on results. Finally, we will formulate profitable trading strategies in the context of the high-volume return premium hypothesis.

The remainder of this article proceeds as follows. In the next part of the paper some results concerning the relation between trading volume and returns from the literature are summarised. Next the dataset is presented and then there is a summation of the simplest version of event study methodology, the main research tool. In the main part of the paper the empirical results for WSE, LSE, FSE and VSE are presented, compared and analyzed. A short summary of the most important issues concludes the paper.

2. Literature overview

In recent decades, a considerable number of papers have been published which examine the role of trading volume in return formation within a theoretical framework. There are many contributions concerning the relation between returns and trading volume for indexes and individual companies.

In the financial literature stock prices are frequently assumed to be noisy. Be that as it may, some recent studies by Blume et al. (1994) and Suominen (2001) argue that data on trading volume convey unique information to the market not contained in prices. This model assumes that informed traders reveal their private information to the market through trades and uninformed traders learn from volume data about the precision and dispersion of informational signals. Hence, return volatility and trading volume exhibit time persistence even when information arrivals do not. Suominen (2001) develops a market microstructure model in which trading volume is used by uninformed traders as a signal of private information in the market which can therefore help to overcome information asymmetries. A common conclusion from these models is that trading volume not only describes market behaviour but actually affects it, since it directly enters into the decision process of market participants. In this sense a strong relationship (contemporaneous as well as causal) between volume and return volatility is suggested.

On the assumption that capital markets are informationally efficient the short term tendency to continue returns, the long-term tendency to return to the equilibrium level (*mean reverting process*) or the role of trading volume in return forecasts and vice versa are interesting phenomena which need explanation. Some authors like deBondt and Thaler (1985), (1987) i (1990), Cambell et al.(1993), Conrad et al. (1994) and Hong and Stein (1999) show, with empirical evidence that negative autocorrelation is the result of initial overreaction to new information which is important to market participants.

In the financial literature there is a dominant view that returns influence trading volume. (i.e. Chen et al. (2001), Lee and Rui (2002), Hiemstra and Jones (1994)). The reverse effect, especially a linear one is less common. The linear impact of trading volume on returns is reported for emerging markets, while in the case of well developed stock markets this relation is rather nonlinear- trading volume has an impact on return volatility. There are some exceptions to this general rule. Here, investigations concerning the anomalous behavior of trading volume processes are of practical importance. These processes may feed into the investor's decision making.

In order to investigate the price or the long-run tendency of returns to achieve long-run equilibrium positions a long horizon of investigations has to be assured. In the short-term after positive (negative) returns often there are also positive (negative) returns in consecutive days.

These, short term observations have been mentioned e.g. in Chan et al.(2000). Some other authors like Chincarini and Lorente-Alvarez (1999) and Llorente et al. (2002), Lee and Swaminathan (2000) or Connolly and Stivers (2003), see these temporary effects as connected with different factors like trading volume. The last variable (trading volume) has played a key role in investment strategies on stock markets analysed by contributors.

A very important recent strand of research concerns so called *overconfidence*. This research interest belongs to behavioral finance, i.e. the psychology of financial markets (Glaser and Weber (2005a), Dreman and Lufkin (2000), Huddart et al. (2005)). The main assumption of this theory is that investors are overconfident about the precision of their information. According to this theory there are two groups of investors: rational and overconfident (irrational) investors. Taking into account psychological factors Daniel et al. (2001) and Odean (1998) assume that irrational investors exhibit their own kind of behavior on capital markets: they buy and sell more than rational investors, they trade more aggressively than rational investors which increases return volatility, they overreact in the light of their private information, which is a reason for wrong stock pricing, although finally any anomalous results which follow from such behavior are compensated in the long-run.

Apart from the impact of trading volume on returns (Aggarwal and Sun (2003)), there is also an opposite effect, known in the financial literature as *the disposition effect,* and this has been investigated. Within the framework of this theory it is assumed that investors sell stocks relatively easily if prices rise taking the latest returns as a benchmark. On the other hand investors postpone selling in the case of negative returns (they wait for a price increase). As in the case of overconfidence theory, a significant positive relation between positive returns and trading volume can be inferred from the disposition effect.

The last two effects can overlap i.e. high trading volume can result from investors overconfidence in their information and also from high positive returns in the preceding days. Discriminating between these two effects is not an easy task. In recent years some researchers, i.e. Statman et al. (2004), Glaser and Weber (2005b) or Chuang and Lee (2006), have been concerned with the conjecture of overconfidence in investors' own information. The authors `checked the performance of some thousands of portfolios from the individual investors' point of view.

An essential contribution concerning the role of trading volume on stock markets is that of Gervais et al. (2001). According to the results presented by the authors extremely a high (small) trading volume is the reason for high (low) returns on the following days when returns for average ("normal") trading activity on a particular stock market are taken as a benchmark. This phenomenon is called by the authors *the high-volume return premium*.

However, on some stock markets not only high, but also low returns on days with increased trading activity are observed and reported. This might result from the investors' tendency to remove from their portfolios stocks whose prices dramatically go down.

According to Gervais et al. (2001) the reason for a high-volume return premium may be some reappraisal of the company. Unexpected information about an increase in trading volume attracts new investors who in the past ignored the company, which would explain the increase in potential buyers of its equities. However, it is the limited number of equities on the market, which causes their price to increase. The authors prove that the impact of extremely high trading volume on returns does not depend on other factors like stock prices, dividend announcements, earning announcements or stock liquidity. For further results on the impact of extremely high trading volume on returns see e.g Kaniel et al. (2003), (2005), Aggarwal and Sun (2003), Lei and Li (2006) and McMillan (2007).

The application of trading volume as a predictor of future prices was demonstrated in an early paper by Ying (1966). This contributor showed that over a period of 6 years an increase (decrease) in daily trading volume on the New York Stock Exchange usually cause a rise (fall) in a level of the S&P 500 Composite Index. This type of investigation was significantly extended by Gervais et al. (2001). The most recent contributors have performed their analysis for individual stocks over 30 years. The authors put forward new explanations of the results, and prove their statistical and economic significance.

Although our paper refers mainly to Gervais et al. (2001) contribution it exhibits significant differences not only in the formulation of some hypotheses and trading strategies but also in the use of different methodology. Moreover, in this paper we also examine the high-volume return premium on emerging markets. A short description of the database applied is the subject of the next section.

3. Dataset

All computations were performed for daily continuous returns at close of trading and the daily trading volume of companies listed on Warsaw Stock Exchange, London Stock Exchange, Frankfurt Stock Exchange and Vienna Stock Exchange. Trading volume is the number of shares traded. The data covers the period between 2nd January 2001 and 28th September 2007. We stress that we did not take into account more recent data in order to exclude the confounding events connected with the outbreak of the word financial crisis and the onset of the rapid fall of stock prices on all stock markets. The WSE is represented by 73 companies listed continuously – but not necessarily from the prime segment – in the whole period. In this group of companies there are also companies belonging to the NFI which has been listed for the long time period separately. As an approximation of the market portfolio we applied the WIG which is the main index of WSE. The Vienna Stock Exchange is represented by 31 companies which were in the ATX or ATX Prime indices over the whole considered period. In the same way 28 companies from the DAX index (FSE) and 81 companies from the FTSE100 index (LSE) were chosen. The market portfolios for these stock exchanges are represented by ATX Prime, DAX and FTSE100 respectively. The data come from the official quotations of WSE, Reuters, Deutsche Börse and Wiener Börse.

Trading volume data characterize high values of skewness and kurtosis, so in order to reduce them a logarithm transformation was applied. However, in the following text for the sake of brevity instead of log-volume we still use the notion of volume (trading volume).

In order to investigate the impact of extreme values of trading volume on stock returns event study methodology is applied. In our opinion it is an appropriate tool in this context, because it is easy to define an event as the occurrence of extremely high or low value of trading volume. In the financial literature there are different versions of event study methodology. These versions are characterised by their definitions of event, event window, estimation window, quality and feature of data, presence or lack of convolution events and so on. The test exercises by Gurgul et al.(2003) convinced us that results from the application of more sophisticated models based on GARCH methodology display only insignificant differences from the results computed by means of e.g. market model. The disadvantage of more advanced models is the necessity of assuring a large size of estimation window which leads to a reduction in the event sample size and make it more difficult to avoid convolution events which can bias the analysis. In order to avoid these problems we apply the simplest version of the event study approach based on the market model. A short description of this version of event study is the subject of the following section.

4. Event study Methodology

We perform our investigations by means of event study methodology. In our contribution we will establish the impact of extreme (very small or very large) values of trading volume on returns. Therefore by definition an event is the occurrence of extremely low or extremely high trading volume. Our definition is similar to the definition given in Gervais et al. (2001). A company exhibits extremely high (low) trading volume on a trading day if this volume is greater (less) than its trading volume as noted in the 50 trading days before and if over this period there was no occurrence of very high or very low trading volume, as at present defined.

The last assumption is a necessary condition for the application of event study methodology. It guaranties that events are separated one from the other and that they are not influenced by other such events. The main problem in event study investigations is to avoid any impact on results by what are called confounding events (the investigated event should be isolated from other events). It is easy to imagine that a company on one day exhibits very high trading volume and on the next day even greater. However, we designate the first day as the event day. The introduction of the nonoverlaping event condition in the definition of the event day (introduced to avoid confounding effects) dramatically reduces the sample size of events, e.g. on WSE from 2822 to 258 in the case of extremely high trading volume and from 3039 to 317 in the case of extremely low trading volume.

The applied event definition allows us to investigate separately the impact of low and high trading volume on returns. This is important in the light of Gervais et al. (2001) investigations. According to their results this impact is completely different. The extremely high trading volume causes positive stock returns and extremely low trading volume implies negative stock returns in the following trading days.

In order to prove the impact of extreme trading volume on the stock returns of a company event study methodology was applied. As an estimation window we chose data from 50 trading days numbered as t=-52, ..., -3 in relation to the event day denoted by t=0. As event window we selected data for t=-2, ..., 5. On the basis of data from individual stock returns R_t and market returns M_t approximated by the WIG coming from window before event (i.e. for t=-52, ..., 5), parameters α and β of the market model

$$AR_t = R_t - \alpha - \beta M_t \tag{1}$$

were estimated.

The market model is the most frequently used model in the framework of event study. By means of formula (1) we calculated abnormal returns AR_t in the estimation window and in the event window Next, for both groups of events i.e. extremely high and extremely low trading volume, we computed for estimation window and the event window mean cross-sectional abnormal returns $A\overline{R}_t$:

$$A\overline{R}_t = \frac{1}{N}\sum_{i=1}^{N} AR_{i,t} \ , \tag{2}$$

where N stands for the number of events in each cluster. The sample standard deviations of mean abnormal returns can be computed by the formula:

$$\hat{\sigma}\left[A\overline{R}_t\right] = \sqrt{\frac{1}{49}\sum_{t=-52}^{-3}\left(A\overline{R}_t - A\overline{\overline{R}}_t\right)^2} \ , \tag{3}$$

where $A\overline{\overline{R}}_t = \dfrac{1}{50}\sum_{t=-52}^{-3} A\overline{R}_t$ is the mean abnormal return in the event window. In order to test the null hypothesis about the absence of event impact on returns t-statistics were calculated.

$$t_R = \frac{A\overline{R}_t}{\hat{\sigma}\left[A\overline{R}_t\right]}. \tag{4}$$

Assuming that mean abnormal returns are normally distributed the statistic given by (4) is t-Student distributed with N-1 degrees of freedom.

By means of event study methodology we studied the impact of extreme trading volume on stock returns (and other relations) for the given stock markets. The empirical results and their analysis is given in the fourth section.

5. Empirical Results

The computation and test results of abnormal returns for the considered stock markets are presented in the tables 1–4. These tables confirm the presence of a high-volume return premium in the case of companies listed on the Warsaw Stock Exchange. The abnormal returns for extremely high trading volume are positive and significant on the pre event day, on the event day and the two following days. This shows the significant impact of extremely high volume on stock returns on the event day and it demonstrates that it holds true for the following days. However, it is slightly difficult to interpret the significantly positive mean abnormal return on the day before the event. This may mean that the reason for the extremely high trading volume on the event day was an unexpected increase in stock prices and that this resulted in positive abnormal returns. However, this fact may be simply evidence for increased interest in a company, which leads to a growth in both returns and trading volume.

The analytical results of abnormal returns for extremely low trading volume do not confirm the high-volume return premium hypothesis. They are all negative in the event window but on the other hand they are insignificant except for the event day and the fourth day after it. This implies an almost zero impact of low volume on returns. The significance of the mean abnormal return on the event day does at least show the existence of a relationship between trading volume and returns but it does not show its direction. Thus it can be concluded that extremely low trading volume does not affect stock returns on WSE.

Table 1. Daily mean abnormal returns for companies listed on WSE in the years 2001–2007

Day t	Extreme high trading volume (258 events)			Extreme low trading volume (317 events)		
	$\overline{AR_t}$ (in %)	*t-Stat*	*p*-value	$\overline{AR_t}$ (in %)	*t*-Stat.	*p*-value
-2	0,024	0,173	0,863	0,027	0,229	0,819
-1	0,409**	2,896	0,004	-0,211	-1,805	0,072
0	2,212**	15,663	0	-0,232*	-1,993	0,047
1	0,763**	5,402	0	-0,161	-1,381	0,168
2	0,337*	2,389	0,018	-0,047	-0,405	0,686
3	0,212	1,504	0,134	-0,221	-1,895	0,059
4	0,251	1,777	0,077	-0,283*	-2,425	0,016
5	-0,234	-1,654	0,099	-0,024	-0,205	0,838

* *significant at 5%.* ** *significant at 1%.*

Table 2. Daily mean abnormal returns in the years 2001–2007 of companies listed on the ATX and ATX Prime in the whole considered period

Day t	Extreme high trading volume (117events)			Extreme low trading volume (109 events)		
	$\overline{AR_t}$ (in %)	*t*-Stat.	*p*-value	$\overline{AR_t}$ (in %)	*t*-Stat.	*p*-value
-2	0,009	0,074	0,942	-0,075	-0,52	0,604
-1	-0,02	-0,159	0,874	-0,08	-0,555	0,58
0	0,379**	3,031	0,003	-0,191	-1,317	0,191
1	0,143	1,145	0,254	-0,133	-0,92	0,359
2	0,086	0,689	0,492	0,259	1,783	0,077
3	0,148	1,179	0,241	-0,201	-1,389	0,168
4	0,026	0,206	0,837	0,109	0,752	0,454
5	-0,035	-0,279	0,781	-0,036	-0,247	0,806

* *significant at 5%.* ** *significant at 1%.*

While there is evidence for it on Warsaw Stock Exchange the results for other markets do not exactly confirm the high-volume return premium hypothesis. It can be noted that significant positive abnormal returns on these stock exchanges occur exclusively on the event day, i.e. on the day of extremely high trading volume. On the remaining days of the event window they are insignificant. Moreover the abnormal returns on the days following the event day exhibit different signs. In the case of the Vienna Stock Exchange the mean abnormal return on the day after the event is positive. However for the London and

Frankfurt Stock Exchanges this value is negative. Although the impact of high trading volume for the Vienna Stock Exchange is statistically insignificant, the positive signs of returns can be treated as some support for a weak form of the high-volume return premium. In this case it can be assumed that investors attracted by increased trading volume sell their equities on the following day. We can expect this effect to be a temporary one.

Table 3. Daily mean abnormal returns in the years 2001–2007 of companies listed on the DAX 30 in the whole considered period

Day t	Extreme high trading volume (130 events)			Extreme low trading volume (83 events)		
	$\overline{AR_t}$ (in %)	t-Stat.	p-value	$\overline{AR_t}$ (in %)	t-Stat.	p-value
-2	-0,059	-0,524	0,601	-0,214	-1,16	0,249
-1	-0,012	-0,106	0,916	0,098	0,531	0,597
0	0,074	0,654	0,514	-0,078	-0,422	0,674
1	-0,051	-0,45	0,654	-0,065	-0,352	0,726
2	-0,112	-0,99	0,324	0,134	0,728	0,469
3	-0,262*	-2,312	0,022	-0,139	-0,75	0,455
4	-0,286*	-2,531	0,013	0,227	1,229	0,223
5	0,013	0,113	0,911	0,245	1,329	0,188

* *significant at 5%.* ** *significant at 1%.*

Table 4. Daily mean abnormal returns in the years 2001–2007 of companies listed on the FTSE 100 in the whole considered period

Day t	Extreme high trading volume (335 events)			Extreme low trading volume (344 events)		
	$\overline{AR_t}$ (in %)	t-Stat.	p-value	$\overline{AR_t}$ (in %)	t-Stat.	p-value
-2	0,168	1,543	0,124	-0,067	-0,796	0,427
-1	0,086	0,794	0,427	-0,01	-0,122	0,903
0	0,349**	3,204	0,001	-0,13	-1,545	0,123
1	-0,057	-0,524	0,601	0,068	0,817	0,415
2	-0,072	-0,662	0,508	-0,04	-0,483	0,63
3	0,212	1,952	0,052	0,045	0,538	0,591
4	-0,074	-0,68	0,497	-0,076	-0,909	0,364
5	0,049	0,448	0,654	-0,015	-0,177	0,86

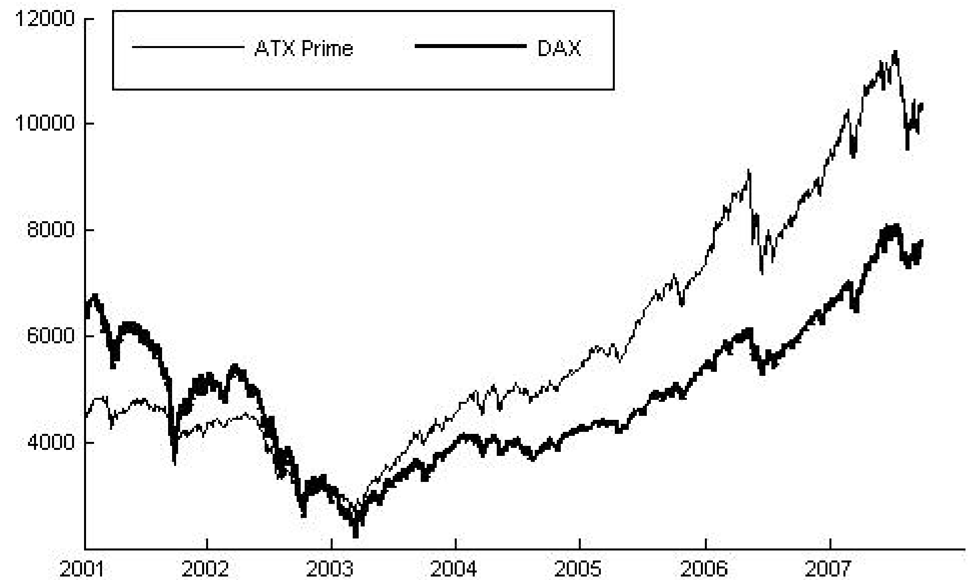
* *significant at 5%.* ** *- significant at 1%.*

In the second group of events, i.e. the group of extremely low trading volume for stocks from the LSE, the FSE and VSE, we did not observe statistically significant abnormal returns even on the event day and on the following days. Moreover, abnormal returns in the event window have different signs which makes any interpretation very difficult. Neither is an explanation of this fact from an economic point of view easy. In the case of a cluster with low trading volume the signs of mean abnormal returns are diversified. However, their lack of significance does not allow clear and certain conclusions. Only one conclusion can be drawn from the test results for extremely low trading volume – such an event has almost no impact on stock returns on following days.

Although the results summarized in tables 1–4 only partially support the high-volume return premium hypothesis, they lead to interesting new questions. The most important question concerns the differences between results for the WSE and other considered stock markets. A highly significant aspect of this is probably the sample of companies taken into account. In the case of the London Stock Exchange, the Frankfurt Stock Exchange and the Vienna Stock Exchange the data concerns blue chips. According to Gervais et al. (2001) the high-volume return premium depends on company capitalization. Normally the high-volume return premium tends to diminish with the size of a company. In the light of the hypothesis of the high-volume return premium the effect would therefore be insignificant because blue chip stocks dominate these markets. In the case of blue chips high trading volume is not a reason for increased interest because they are under the continual scrutiny of market analysts, the mass media, investors, and especially institutional investors. Therefore the public has a lot of information concerning blue chips, and the size of trading volume does not attract attention. It is not the only, or even most important factor determining investor behavior.

In the case of the WSE the data are not restricted exclusively to one segment of the market (e.g. blue chips). The sole criterion is the permanent presence of a company on the market in a given time period. Therefore in the considered cluster of companies are not only blue chips like PEKAO, TP S.A., Bank BPH, but also small companies like ATLANTIS or 01NFI.

Graph 1. ATX Prime and DAX values from January 2001 to September 2007.



It can be seen that the considered period (from 2001 to 2007) falls into two periods in which stock market development was different. This is reflected as an example in graph 1 for the ATX Prime and the DAX. Up until about March 2003 the world markets were bear markets. Therefore the indexes tended to fall. Subsequently, from April 2003, all considered stock indexes gradually increased (bull market till September 2007). In our research we naturally ventured that the variability of stock market behavior in the considered period might have a significant impact on the size of the high-volume return premium. In order to test this, we repeated all former computations for the data from the bull market period i.e. from the 1st of April 2003 till 28th of September 2007. The obtained results are presented in the tables 5–7. For the London Stock Exchange and the Vienna Stock Exchange extremely high trading volume lead to mean abnormal returns on the event day and on the next day being significant. We can draw from this the conclusion that our conjecture about the high-volume return premium holds true for these stock exchanges when there is a bull market period. The results concerning the Frankfurt Stock Exchange are not quite clear since mean abnormal returns are positive and significant on the event day and on the fourth day after the event day. One day after the event the mean abnormal return becomes negative, which does not occur in the case of the London and Vienna Stock Exchanges. In the case of extremely low trading volume we can not find any significant dependencies between trading volume and returns.

Table 5. Daily mean abnormal returns in the period April 2003–September 2007 of companies listed on ATX Prime and ATX in 2001–2007

Day t	Extreme high trading volume (79 events)			Extreme low trading volume (71 events)		
	$\overline{AR_t}$ (in %)	t-Stat.	p-value	$\overline{AR_t}$ (in %)	t-Stat.	p-value
-2	-0,084	-0,426	0,671	-0,269	-1,303	0,197
-1	-0,253	-1,286	0,202	-0,163	-0,79	0,432
0	0,598**	3,043	0,003	-0,143	-0,692	0,491
1	0,495*	2,52	0,014	-0,17	-0,821	0,414
2	0,094	0,479	0,633	0,264	1,275	0,206
3	-0,005	-0,027	0,978	-0,121	-0,584	0,561
4	0,069	0,351	0,726	0,112	0,539	0,591
5	-0,124	-0,631	0,53	0,028	0,137	0,891

* *significant at 5%.* ** *significant at 1%.*

Table 6. Daily mean abnormal returns in the period April 2003–September 2007 of companies listed on DAX in 2001–2007

Day t	Extreme high trading volume (79 events)			Extreme low trading volume (71 events)		
	$\overline{AR_t}$ (in %)	t-Stat.	p-value	$\overline{AR_t}$ (in %)	t-Stat.	p-value
-2	-0,084	-0,426	0,671	-0,269	-1,303	0,197
-1	-0,253	-1,286	0,202	-0,163	-0,79	0,432
0	0,598**	3,043	0,003	-0,143	-0,692	0,491
1	0,495*	2,52	0,014	-0,17	-0,821	0,414
2	0,094	0,479	0,633	0,264	1,275	0,206
3	-0,005	-0,027	0,978	-0,121	-0,584	0,561
4	0,069	0,351	0,726	0,112	0,539	0,591
5	-0,124	-0,631	0,53	0,028	0,137	0,891

* *significant at 5%.* ** *significant at 1%.*

In order to test the dynamic properties of any high-volume return premium investigations were performed in approximately two year windows shifted in blocks of 60 trading days. The first window contained the first 500 returns plus trading volume data. The second window was derived from the first by moving it forward by 60 trading days i.e. approximately 3 months. The third window was obtained from the second by moving it forward by the next 60 and so on. Thus, the first computation concerned the period from January 2001 to the end of December 2002, the next was performed for the period April 2001– March 2003, etc. In this way we performed for every market 21 tests for the existence of the high-volume return premium.

Table 7. Daily mean abnormal returns in the period April 2003–September 2007 of companies listed on FTSE100 in 2001–2007

Day t	Extreme high trading volume (221 events)			Extreme low trading volume (248 events)		
	$\overline{AR_t}$ (in %)	*t-Stat.*	*p*-value	$\overline{AR_t}$ (in %)	*t-Stat.*	*p*-value
-2	0,170*	2,202	0,029	-0,036	-0,403	0,687
-1	-0,013	-0,173	0,863	-0,098	-1,093	0,275
0	0,572**	7,416	0	-0,155	-1,725	0,086
1	0,248**	3,209	0,002	-0,064	-0,717	0,474
2	-0,093	-1,208	0,228	0,032	0,36	0,719
3	-0,042	-0,546	0,586	-0,048	-0,537	0,592
4	-0,023	-0,293	0,77	-0,137	-1,53	0,127
5	-0,015	-0,189	0,85	0,064	0,71	0,478

* *significant at 5%.* ** *significant at 1%.*

The results are summarized in tables 8–10. These results confirm that in the case of companies listed on FTSE100 the high-volume return premium is relatively pronounced for the bull market. We can see, that on the whole for the bull market the mean abnormal returns are positive and significant on an event day (t=0) and one day after (t=1). This observation applies to the subperiods from 26th of November 2002 to 24th of April 2006. The results within this dynamic investigation do not confirm the high-volume premium hypothesis for companies listed on the Frankfurt and Vienna Stock Exchanges.

The presented results indicate that the high-volume return premium is present in the case of companies listed not only on the Warsaw Stock Exchange but also for companies listed on the largest European stock exchange – the London Stock Exchange but only if there is a bull market.

From the above calculations the question arises to whether on the basis of the high-volume return premium hypothesis a profitable investment strategy can be based. Possible strategies will be considered only in the case of extremely high trading volume.

In order to formulate and examine the strategies two types of daily returns at close will be used: daily returns

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{5}$$

and returns in relation to price P_0 on the event day:

$$\widetilde{R}_t = \frac{P_t - P_0}{P_0} \qquad (6)$$

Table 8. Daily mean abnormal returns (in %) for extreme high trading volume in the subperiods from January 2001–September 2007 for companies listed on ATX and ATX Prime

Subperiods		Events	Mean abnormal returns						
			t=-2	*t*=-1	*t*=0	*t*=1	*t*=2	*t*=3	*t*=4
03.01.01	19.12.02	39	0,18	0,18	-0,05	-0,31	-0,2	0,35	0,12
28.03.01	20.03.03	39	0,26	0,14	0,64*	-0,16	-0,04	0,25	0,45
25.06.01	17.06.03	56	-0,22	-0,36	0,28	-0,39	-0,05	0,16	0,82**
17.09.01	09.09.03	48	0,08	0,2	0,28	-0,05	-0,14	-0,08	0,07
10.12.01	02.12.03	44	0,55*	0,46	0,72**	0,15	-0,15	0,08	-0,19
11.03.02	02.03.04	48	-0,18	0,21	0,53*	0,35	-0,2	0,05	-0,16
06.06.02	27.05.04	43	-0,16	0,05	0,39	0,13	0,04	0,35	-0,11
29.08.02	19.08.04	45	-0,4	-0,09	-1,18**	0,46	0,52	0,13	-1,49**
21.11.02	11.11.04	35	-0,24	0,15	1,12**	0,32	-0,01	0,02	-0,19
20.02.03	07.02.05	41	-0,26	0,24	0,58*	0,02	0,07	-0,06	-0,01
20.05.03	04.05.05	35	-0,60*	0,12	0,90**	1,07**	0,66**	-0,3	-0,80**
12.08.03	27.07.05	35	-0,13	0,18	0,51**	0,09	0,63**	-0,03	-0,58**
04.11.03	19.10.05	33	-0,51	-0,08	1,48**	0,37	0,36	-0,2	-0,43
03.02.04	12.01.06	35	-0,12	-0,32	0,03	0,50**	0,39	-0,45	-0,26
29.04.04	06.04.06	34	0	-0,6	0,74*	0,14	0,77*	-0,42	-0,25
22.07.04	04.07.06	44	-0,03	-0,27	0,45	0,18	0,31	0,15	-0,17
14.10.04	26.09.06	41	0,21	-0,28	0,98**	0,44	0,13	0,31	-0,71**
10.01.05	19.12.06	26	0,37	-0,56	0,56	0	-0,22	0,3	0,22
06.04.05	16.03.07	41	0,51*	-0,23	1,60**	0,69**	-0,32	0,06	0,04
29.06.05	14.06.07	34	0,3	-0,31	1,29**	-0,06	-0,43	0,36	0,37
21.09.05	06.09.07	41	0,22	0,36	1,61**	0,93**	0,27	1,04**	-0,33

* *significant at 5%.* ** *significant at 1%.*

Formula (6) means that the price on day *t* is compared to the closing price on the event day. This should help in choosing the best day to sell stock bought on the event day. The following strategies are considered: when there is extremely high trading volume an investor buys a stock at close price. Next he can sell the stock of a company at the closing price of the first day after the event (Strategy I) or on the second day (Strategy II), the third day (Strategy III) or the fourth day (Strategy IV) after the event. Next the investor should wait until the next event for any stock, and then the above procedure should be repeated. For these strategies it is assumed that all transactions are cost-free.

We set up a procedure to check the profitability of the above strategies by computing the final value of 1 Zloty invested in the case of WSE or £ 1 in the case of the LSE. Because the investor has only the initial capital which is constantly invested he can not invest it for all events. If for example on a certain day there is extremely high trading volume for more than one stock, say *k* stocks, then it should be decided that the capital is invested only in one stock or else the investor can spread it. We assume that in this situation the market participant invests the same portion of his money in all these stocks. We additionally assume that the investor invests all his money on each occasion, so that after buying stocks he does not have ready cash to make new investments. Thus, the buying of new stocks (as a result of a new event) cannot be performed on the same day as the selling of stocks bought based on the former event day. This assumption is realistic, because sell-buy transactions can not be performed during the same session at close. It is not possible because the investor does not know the selling price of stocks bought on the former event day and thus he does not know how much he can invest in new stocks.

Table 9. Daily mean abnormal returns (in %) for extreme high trading volume in the subperiods from January 2001–September 2007 for companies listed on DAX

Subperiods		Events	Mean abnormal returns						
			t=-2	*t*=-1	*t*=0	*t*=1	*t*=2	*t*=3	*t*=4
02.01.01	18.12.02	49	-0,58*	-0,35	-0,64*	-0,21	0,05	-0,49	-0,54*
27.03.01	19.03.03	50	-0,32	-0,41	-0,56*	0,06	0,12	-0,18	-0,06
22.06.01	16.06.03	44	0,08	-0,80**	-0,67*	0,33	0,29	-0,98**	0,11
14.09.01	08.09.03	31	0,4	0,33	0,71	-0,01	0,75	-0,13	0,44
07.12.01	01.12.03	34	0,28	-0,11	1,46**	-0,18	0,13	0,02	0,14
08.03.02	01.03.04	45	0,27	0,04	0,24	0,12	-0,05	0,04	-0,58*
05.06.02	26.05.04	25	0,80*	-0,01	1,09**	0,1	0,07	-0,43	0,12
28.08.02	18.08.04	24	0,93*	-0,04	0,75*	-0,11	-0,38	-0,02	0,38
20.11.02	10.11.04	49	0,19	0,12	0,25	0,06	-0,18	-0,18	0,05
19.02.03	04.02.05	30	0,09	0,16	0,33	-0,17	-0,27	0,02	-0,04
19.05.03	03.05.05	35	0,3	0,26	0,63**	0,09	-0,08	-0,09	-0,12
11.08.03	26.07.05	31	0,09	0,27	0,05	-0,18	0,05	0,09	-0,01
03.11.03	18.10.05	41	0,11	0,11	0,77**	-0,17	-0,12	-0,16	-0,13
02.02.04	11.01.06	30	0,24	0,37*	0,21	-0,07	0	0,14	-0,31
28.04.04	05.04.06	42	-0,14	0,22*	0,35**	0	-0,02	0,11	-0,16
21.07.04	03.07.06	44	-0,04	-0,07	-0,05	0,05	-0,09	-0,1	-0,32**
13.10.04	25.09.06	26	-0,21	0,34*	-0,21	-0,02	-0,11	-0,05	-0,44**
07.01.05	18.12.06	41	-0,17	0	-0,45**	0,09	-0,05	-0,17	-0,32*
05.04.05	15.03.07	52	0,03	0,17	0,36**	-0,01	-0,1	-0,12	-0,31**
28.06.05	13.06.07	50	0,08	0,16	0,36**	-0,04	-0,19	-0,15	-0,35**
20.09.05	05.09.07	42	0,05	0,06	0,35*	0,03	-0,15	-0,32*	-0,18

* *significant at 5%.* ** *significant at 1%.*

Moreover, because the investor invests all his funds he can buy new stocks only after selling the old ones. Therefore in the case of Strategy I, if stocks are bought on an event day and sold the next day, the investor can buy new stocks only on the second day after the previous event. This means that any new event immediately following the event under study will be ignored. In the case of Strategy II the investment does not depend on events which took place during the two days after buying the stocks, because according to this strategy the stocks are sold on the second day after buying. The same holds for Strategy III and Strategy IV where the investment according to these strategies does not depend on potential events on the third or fourth day after the previous event.

Table 10. Daily mean abnormal returns (in %) for extreme high trading volume in the subperiods from January 2001–September 2007 for companies listed on FTSE100

Subperiods		Events	Mean abnormal returns						
			t=-2	t=-1	t=0	t=1	t=2	t=3	t=4
03.01.01	24.12.02	132	0,1	0,36	0,39	-0,17	-0,06	0,41	-0,28
28.03.01	21.03.03	120	0,08	0,33	0,2	-0,27	-0,08	0,1	-0,18
26.06.01	19.06.03	142	-0,02	0,15	-0,26	-0,26	-0,36**	0,36	-0,15
19.09.01	12.09.03	93	-0,13	0,17	0,51*	-0,75**	0,07	0,35	-0,28
12.12.01	05.12.03	122	-0,02	0,08	0,14	-0,29	0,23	0,24	-0,27
11.03.02	03.03.04	105	0,09	0,03	-0,41*	-0,42*	0,47*	0,44*	-0,24
10.06.02	01.06.04	85	0,09	-0,02	0,65**	-0,18	-0,05	0,44*	0,07
03.09.02	24.08.04	84	0,29	0,01	-0,88**	0,08	0,04	0,23	0,09
26.11.02	17.11.04	107	0,1	-0,30*	-1,14**	0,44**	0,28*	0,27*	-0,2
21.02.03	14.02.05	97	0,32*	-0,04	0,79**	0,36**	0,03	0,02	-0,19
21.05.03	12.05.05	97	0,30*	-0,19	0,47**	0,56**	-0,03	-0,24*	-0,23
14.08.03	05.08.05	84	0,28*	-0,17	0,58**	0,43**	-0,02	0,09	0
07.11.03	31.10.05	119	0,03	-0,01	0,67**	0,28*	-0,01	0,1	-0,13
04.02.04	26.01.06	88	0,14	0	0,28*	0,40**	-0,16	0,08	-0,05
30.04.04	24.04.06	94	-0,02	0,01	0,26*	0,30*	0,08	-0,09	0
27.07.04	19.07.06	112	-0,01	-0,01	0,18	-0,03	-0,15	0,12	0,16
20.10.04	12.10.06	141	0,03	0,28**	0,23*	0,03	-0,21*	0,06	0,07
17.01.05	09.01.07	99	-0,02	0,04	0,23*	-0,09	-0,12	0,11	0,07
13.04.05	03.04.07	117	0,1	0,11	0,38**	0,07	-0,12	0,01	0,05
08.07.05	02.07.07	128	0,14	0,06	0,21*	-0,08	-0,18	-0,05	0,04
03.10.05	25.09.07	91	0,14	-0,11	0,70**	-0,14	-0,19	-0,03	0,13

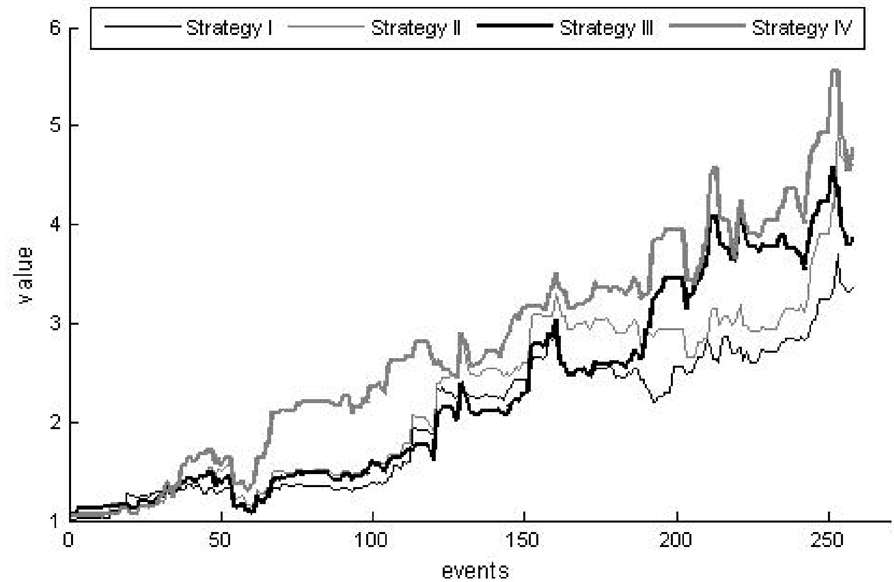
* *significant at 5%.* ** *significant at 1%.*

The final value of the initially invested 1 Zloty or £ 1 can be calculated by the formula

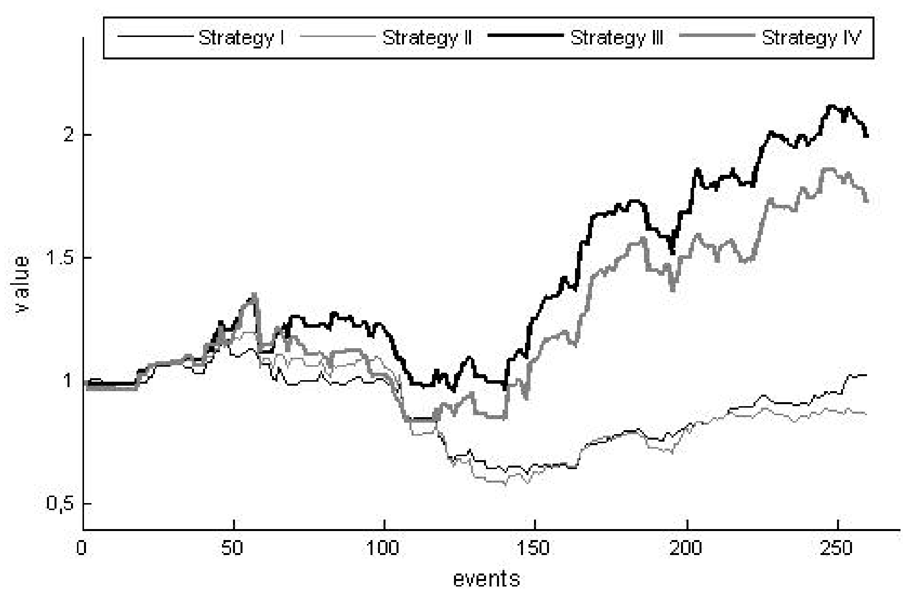
$$K = \prod_{t=1}^{N_j} \left(1 + r_t\right),$$

where N_i stands for the number of all investments in the case of the i-th strategy (i=1…4) and r_t stands for the return on this investment. Graphs 2 and 3 present the results applying each strategy on the basis of datasets and events described in earlier sections of this paper for the WSE and the LSE.

Graph 2. Results of applying strategies based on the high-volume return premium hypothesis for companies listed on WSE in the years 2001–2007.



Graph 3. Results of applications of strategies based on high-volume return premium hypothesis for companies listed on FTSE100 in the years 2001–2007.



From graph 2 it can be seen that all investment strategies applied for stocks from the WSE are profitable during the whole considered period. One can see that investment according to Strategy IV gives the best results, i.e. selling of stocks on the fourth day after the event. In this case each Zloty invested at the beginning of 2001 is worth 4.78 at the end of September 2007, which gives a 378% rate of return. Even the "worst" strategy gives a return of 236%. In comparison, strategy IV applied solely to the WIG market index WIG yields a capital increase of 100%.

The profitability of the presented investment strategies on LSE is quite different. While Strategies III and IV yield a profit for almost the whole period, strategies I and II suffer a loss in the long run. The final value of the initial £1 is equal to £ 1.02, £ 0.87, £1.99 and £ 1.73 respectively. Hence, strategy III, the most profitable, gives an almost 100% rate of return. It should be noted that the value of the investment increased mostly in the later period, i.e. on the bull market. This once again confirms our earlier findings about the high-volume premium hypothesis on the LSE.

Under the above assumptions about the applied strategies it should be obvious that when they are applied some events are ignored. The choice of starting point could therefore influence their profitability. However, our investigations with different starting points suggest that this influence is insignificant.

A summary of the main findings is presented in the last section of this contribution.

6. Conclusions

Our research on an emerging market (WSE) and three well developed capital markets, the London Stock Exchange, the Frankfurt Stock Exchange and the Vienna Stock Exchange concerned the role of trading volume in the process of price formation on these markets during the period form January 2001 to September 2007. We tested the high-volume return premium hypothesis. As well as an analysis of the whole period we investigated its dynamic properties by taking into account bull and bear markets and by considering it in subperiods. The basic tool applied in our analysis was a variant of event study.

Our computations were performed for the period from the 2nd of January 2001 to the 28th of September 2007. In the first subperiod (to April 2003) there was a bear market and in the second subperiod there was a bull market. By the end of 2007 the first indications of the crisis appeared. By the end of 2008 the crisis deepened dramatically, so we excluded any more recent data to avoid the bias of these unusual circumstances.

The most important finding concerns the Warsaw Stock Exchange. We showed the existence of a high-volume return premium for stocks from the WSE in the case of extremely high trading volume However, we could not confirm that low trading volume causes low (negative) returns. According to empirical results for the LSE, the FSE and the VSE extremely high trading volume has in general almost no influence on returns. The most probable reason for the lack of any high-volume return premium is the nature of the data under consideration. For these stocks only blue chips were considered whereas the high-volume return premium is more strongly pronounced in the case of small companies. The positive significant effect of high trading volume on returns can be detected in the dynamic windows described above (subperiods of bull market).

Based on the results, i.e. the confirmation of the existence of the high-volume return premium on the WSE and partially on the LSE we considered four investment strategies which proved to be profitable on WSE and partially profitable on LSE.

In the future research to fully check the high-volume return premium hypothesis for all stocks on given markets should be considered any relations and established should be tested for the entire time period of the present crisis.

REFERENCES

AGGARWAL, R. & SUN, M., 2003. Trading volume extremes and the subsequent price behavior. *Financial Management Association.*

BLUME, L. EASLEY, D. & O'HARA, M., 1994. Market statistics and technical analysis: The role of volume. *Journal of Finance*, 49, pp.153–181.

CAMPBELL, J. GROSSMAN, S. & WANG, J., 1993. Trading volume and serial correlation in stock returns. *Quarterly Journal of Economics*, 108, pp.905–939.

CHAN, K. HAMEED, A. & TONG, W., 2000. Profitability of momentum strategies in the international equity markets. *Journal of Financial and Quantitative Analysis*, 35, pp.153–172.

CHEN G., FIRTH M. & RUI O.M., 2001. The Dynamic Relation Between Stock Returns, Trading Volume, and Volatility. *The Financial Review,* 38, pp.153–174.

CHINCARINI, L.B. & LORENTE-ALVAREZ, J.-G., 1999. Volume and Return Information on Individual Stocks. [online] SSRN eLibrary: available at: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=883360

CHUANG, W.-I. & LEE, B.-S., 2006. An empirical evaluation of the overconfidence hypothesis. *Journal of Banking and Finance,* 30, pp.2489–2515.

CONNOLLY, R. & STIVERS, C., 2003. Momentum and reversals in equity-index returns during periods of abnormal turnover and return dispresion. *The Journal of Finance,* 58, pp.1521–1555.

CONRAD, J. HAMEED, A. & NIDEN, C., 1994. Volume and autocovariances in short-horizon individual security returns. *The Journal of Finance*, 49, pp.1305–1329.

DANIEL, K. HIRSHLEIFER, D. & SUBRAHMANYAM, A., 2001. Overconfidence, arbitrage, and equilibrium asset pricing. *Journal of Finance*, 56, pp.921–956.

deBONDT, W. & THALER, R., 1985. Does the stock market overreact? *The Journal of Finance*, 40, pp.793–805.

deBONDT, W. & THALER, R., 1987. Further evidence an investor overreaction and stock market seasonality. *The Journal of Finance*, 42, pp.557–581.

deBONDT, W. & THALER, R., 1990. Do security analysts overreact? *The American Economic Review*, 80, pp.52–57.

DREMAN , D. & LUFKIN, E., 2000. Investor overreaction: Evidence that its basis is psychological. *The Journal of Psychology and Financial Markets*, 1, pp.61–75.

GERLACH, R. CHEN, C. LIN, D. & HUANG, M.-H., 2006. Asymmetric responses of international stock markets to trading volume. Physica A, 360, 422–444.

GERVAIS, S. KANIEL, R. & MINGELGRIN, D., 2001. The high-volume return premium. *The Journal of Finance*, 56, pp.877–919.

GLASER, M. & WEBER, M., 2005a. Overconfidence and trading volume. Technical report, Swedish Institute for Financial Research.

GLASER, M. & WEBER, M., 2005b. Which past returns affect trading volume. Technical report, Swedish Institute for Financial Research.

GURGUL, H. MESTEL, R. & SCHLEICHER, C., 2003. Stock market reactions to dividend announcements: Empirical evidence from the Austrian stock market. *Swiss Society for Financial Market Research*, 17, pp.332–350.

HIEMSTRA, C. JONES, J.D., 1994. Testing for linear and nonlinear Granger causality in the stock price – volume relation. *Journal of Finance,* 49, pp.1639–1664.

HONG, H. & STEIN, J., 1999. A unified theory of underreaction, momentum trading, and overreaction in asset markets. *The Journal of Finance*, 54, pp.2143–2184.

HUDDART, S. YETMAN, M. & LANG, M., 2005. Psychological factors, stock price paths, and trading volume. *AFA 2006 Boston Meetings Paper* [online], available at: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=687065

KANIEL, R. LI, D. & STARKS, L., 2003. The high volume return Premium and the investor recognition hypothesis: International evidence. Working Paper.

KANIEL, R., LI, D. & STARKS, L., 2005. Investor visibility events: Cross-country evidence. Working Paper.

LEE B.S. & RUI O.M., 2002. The dynamic relationship between stock returns and trading volume: Domestic and cross-country evidence. *Journal of Banking and Finance,* 26, pp.51–78.

LEE, C. & SWAMINATHAN, B., 2000. Price momentum and trading volume. *The Journal of Finance*, 55, p.2017–2069.

LEI, A. & LI, H., 2006. Trading volume shocks and subsequent stock returns. http://www.fma.org/SLC/Papers/FMA_TradingVolumeShocks.pdf. [Accessed 12 June 2009]

LLORENTE, G. MICHAELY, R. SAAR, G. & WANG, J., 2002. Dynamic volume-return relation of individual stock. *The Review of Financial Studies*, 15, pp.1005–1047.

MC MILLAN, D., 2007. Non-linear forecasting of stock returns: Does volume help? *International Journal of Forecasting*, 36, pp.115–126.

ODEAN, T., 1998. Volume, volatility, price, and profit when all traders are above average. *Journal of Finance*, 53, pp.1887–1934.

STATMAN, M. THORLEY, S. & VORKINK, K., 2004. Investor overconfidence and trading volume. *AFA 2004 San Diego Meetings,* [online], available at:

http://www.afajof.org/pdfs/2004program/UPDF/P463_Behavioral_Finance.pdf [Accessed 12 June 2009]

SUOMINEN, M., 2001. Trading volume and information revelation in stock markets. *Journal of Financial and Quantitative Analysis*, 36, pp. 545–565.

YING, C.C., 1966. Stock market prices and volumes of sales. *Ecomometrica,* 34, pp.676–686.

STATISTICS IN TRANSITION-new series, July 2009
Vol. 10, No. 1, pp. 153—154

CURRENT ISSUES IN PUBLIC STATISTICS: THE COMMUNICATION AND DISSEMINATION OF OFFICIAL STATISTICS

From the Editor:

The motivation for stressing importance of the above mentioned issues (the communication and dissemination of official statistics) is rooted in variety of needs associated with daily practice of centers responsible for those types of activities. However, there is a special occasion to do it now. Namely, from 13 to 15 May 2009, an international conference – *The UNECE Work Session on Statistical Dissemination and Communication* organized by United Nations Statistical Commission and Economic Commission for Europe – was held in Warsaw, Poland, gathering representatives from 27 countries[1] and from the Statistical Office of the European Communities (Eurostat) and the Organisation for Economic Co-operation and Development (OECD).

Although intention of this note is calling reader's attention to the significance of these issues, taking the opportunity of the meeting devoted to their discussion – without any attempts to provide a coverage of the event itself [2] – some of the major types of concerns (as they were reflected in the meeting agenda) seem worthwhile mentioning here. First of all, it is clear interest in the best practice generating approach, geared to better preparation and delivery of the high quality information products to diverse users and multiple-audience, either directly or indirectly. It is paralleled by the growing awareness of the necessity to prepare also them to better reception of the information provided and to its larger utilization for different purposes. The scope of the discussed ways of information use ranges from expertise and policy design to decision making and evaluation; however, still a room was left for a more discernible presentation of the questions related to meta-information and metadata, especially in the context of the information transmission and as a precondition for proper use of information produced. Next important topic concentrated on the key role played by intermediaries in the producer-user communication channel, with a prominent role

[1] Countries represented at the meeting: Australia, Azerbaijan, Belarus, Belgium, Canada, Czech Republic, Denmark, Finland, Georgia, Germany, Ireland, Italy, Kazakhstan, Lithuania, Luxembourg, Norway, Poland, Russian Federation, Slovakia, Slovenia, Spain, Sweden, Switzerland, United Kingdom and United States of America.

[2] For the meeting report see http://www.unece.org/stats/documents/2009.05.dissemination.htm

of the media – ’working with media’ was entitled one of the main sessions. And finally, it is persistently present concern about respondent and vital importance of cooperation between data provider (subject) and data producer, as a prerequisite of gathering the most valuable information, especially in micro-data generating processes.

Since all the papers presented at the Warsaw’s UNECE Work Session on Statistical Dissemination and Communication are available under the link given in footnote 2, we decided to include in this section only two of them – they both were presented on behalf of the Central Statistical Office of Poland.

Włodzimierz Okrasa
Editor-in-Chief

STATISTICS IN TRANSITION-new series, July 2009
Vol. 10, No. 1, pp. 155—162

THE RELATIONSHIP BETWEEN OFFICIAL STATISTICS AND THE MEDIA IN POLAND – THE ROLE OF INFORMATION SERVICE AND THE CONFRONTATION AND COOPERATION ZONES

Wiesław W. Łagodziński

ABSTRACT

The relationship between official statistics and the media in Poland – including the scope and the timetable of the mutual collaboration on statistical information delivery and access – is regulated by the Law on Official Statistics and the Press Law, with a set of procedures that are rigorously observed in practice. The media are provided with full range of 'publicly available' information being produced within the system of official statistics that remains under the responsibility of the President of the Central Statistical Office. The cooperation with the media is based on a user-friendly and free of charge access to all public data files.

Key words: official statistics, the media, information service.

1. Introduction

Pursuant to the Act of 1995, official statistics provides information services for the wide spectrum of users, including general public and the mass media. This information service, which is often involved in conflicts and confrontation with the media, originates from the fact that the media have become one of the most important factors in shaping political, social, and economic opinions and decisions. Unfortunately, the disproportionate power of the media hardly goes hand-in-hand with the mass media's competence in the area of statistics, including objectivity and impartiality required for professional presentation of the statistical information to a general public and other audience.

During the transition period in Poland, the media service system has been developing as a part of systemic, social and political changes in the country. The first stage, until 1995, entailed the construction of such a system from scratch, as there had previously been hardly existing any independent media or impartial statistical information policy. The second period, from 1995 through 2002, was characterized by a gradual adjusting of the economy to the European standards, as well as the official statistics' presence on the market of economic and social information. Since 2002, we have actually implemented EU standards and met conditions, whereas the information service for the media has taken a form slightly different from those operating in other countries, being a more diversified and oriented towards the needs of the media. However, over the last 2 years new economic conditions, together with newly implemented statistical operations, posed several challenges to the information service system. The information and technological potentials of the Polish media have grown enormously, triggering a tremendous increase in terms of their sense of aggressively manifested public mission. All this has made statistics the object of numerous information-related provocations. In addition, there have appeared certain intra-statistics problems, arising from the subsequent stages of development, including huge expansion of the informational portals and the electronic information systems allowing for introduction of electronic reporting, easily made modification of classification systems, and preparation for the national censuses – the National Agricultural Census (PSR) in 2010, and the National Population Census (NSP) in 2011. This constitutes the background for the issue of the relationship between official statistics and the media discussed below (however, the dramatic legal confrontations that took place between representatives of official statistics and the judiciary administration, due to some misinterpretation of the fundamental principles of "statistical confidentiality, also should be mentioned in this context).

2. Scope and forms of information service for the media -- what content is provided to the media, when and how?

The information service for the media has actually become the main organisational form of the service of all sections of the social audience of statistics. The general principle observed here is **the 3XR principle,** comprising **Equivalence** (in Polish ***Równorzędność***), **Simultaneity** (***Równoczesność***), and **Equality** (***Równoprawność***), showing in what way information is made available to the public.

2.1.Thematic scope and frequency of publications

Basic statistical volumes and values, published in compliance with the legal provisions imposing the publicity obligation on the President of the CSO, are the most significant pieces of information published. These usually include 12–15 presentations, containing from a dozen or so to several dozen pieces of information which concern the principal areas and key phenomena for running the country. The data included refer to inflation, the labour market, wages and salaries, unemployment, foreign trade, retail trade, manufacture, etc. As regards this group of information, deadlines which cannot usually be exceeded by a part of a second are extremely rigorously kept. The second group consists of final information and information notes (including monitoring data and periodic surveys), concerning various areas of life and economy, such as population, projections, labour market, production of the main products, etc. Aggregated studies, such as tabular "Statistical Bulletin" and descriptive information on the economic situation, are published on a monthly basis, usually in connection with a press conference. Various materials and studies related to the concept of a monthly press conference are also published every month. Thorough studies, both in hard copy and in electronic format, are released on a quarterly, half-yearly, and yearly basis. Finally, all types of statistical yearbooks, both general and related to a specific field of statistics, are published annually. Each of the publications mentioned above is communicated and provided to the media, mostly in electronic format, or in the form of studies published on the information portal, mostly in the press section, but often also in the general one.

2.2.Schedules and deadlines

The programme of information to be presented is published a year, a month, or a week in advance. It is provided to journalists, being made available on the Internet, as well as served for the Government Information Centre, published in the form of publication announcements, and disseminated, whenever there appears any such request or order. The schedules are viewed as fixed legal and information-related obligations, and as regards dates of the presentation – we extremely rarely withdraw from accepted arrangements. Throughout the last 2 years, there have been only two instances of delayed publication dates resulting from the breakdown of our computer system. On the contrary, it very often happens that an attempt is made to accelerate the calculation of statistical indicators published at the end of the month, so as to make it possible for these indicators to be published on the date of the press conference (usually taking place on day 24–25 of the month). We notify our users of any change of the deadline 24 hours in advance, and announce any such change „urbi et orbi".

2.3.Forms of dissemination. Press conference as a specific release form

Materials are released through all possible information carriers, and via all communication channels, the most significant of which include the Internet, e-mail, print-outs, DVD or CD, telephone, fax, briefing, press conferences, and face to face service. As part of the service technology, we have introduced a multifunction device which scans and formats an electronic announcement, transfers it to the PC, and enables it to be sent via the Internet. This makes the time of compiling multi-thematic materials, and the materials covering various reporting periods, far easier and considerably shorter. We make our publications available when ordered, though within reasonable limits. We are prudent in managing publications since this is an expensive form of information supply to journalists. Only in the case of general statistical yearbooks, all the journalists cooperating with statistical bodies receive their own copy, together with a CD version.

Among the release forms, special emphasis is put on the public, open, and publically-accessible press conferences held by the President or Vice-Presidents of the CSO. The press conference is devoted mostly to presenting the socioeconomic results for the preceding month, as well as to a number of materials and data which are significant, urgent and crucial for the situation in the country. These concern unemployment, labour market, wages and salaries, agriculture, housing, etc. The date of the conference is announced a month in advance. The invitations to the conference are sent three times, the last being 1 day before the conference. They are placed on the Internet, and sent to all the media and Government Information Centre. We also individually invite those journalists whose position in the media market is most prominent. The conference is attended by CSO experts and specialists, as well as by CSO managers and heads of particular CSO divisions. The entire conference is recorded in the "audiotele" system (the registered trademark of the Polish Television (TVP), denoting services rendered or commissioned by TVP via the fixed-line telephony – *translator's note*). It is then transmitted to statistical offices, as well as made available for playing back on one of the TV educational channels. The conference takes the form of a presentation of CSO analyses, supported by audiovisual materials. Following the presentation, we usually respond to all questions asked by the attendees. There are no limits to the number of questions, and they are answered immediately without postponing to any subsequent meeting. All conference materials, including sound records, are made available on our information portal in the press tag which is easy to notice, being placed in the most prominent place, right below the portal headline.

The general principle that governs the services we provide to the media entails absolute prioritisation and timely execution of the orders submitted, maximum satisfaction of media expectations, and observance of the 3XR principle mentioned above. The rule of multiplication of materials dissemination to the media takes effect that objections to the lack of reaction to the orders submitted by the media are extremely rare. However, there are certain problems connected with too aggressive and repeated attempts to use identified personal data, or to gain earlier access to the data, as well as to obtain the data for exclusive use. The most recent case (occurring on 31 March) concerned the request for a list of goods and representative items covered in the price survey, as well as the products added and removed from this list. Therefore, it is a general principle that any material prepared at the nominal order of any editorial board can be of secondary use and can be disseminated several times, and that no editorial board can claim the exclusive copyright to put the question, or to be given the response to it. The observance of this media democracy solution seems to create certain difficulties for the Polish mass media. However, the CSO observes the "first-come-first-served" principle as regards responding to questions raised. In the Polish media, the topics raised usually enter public domain via press columns, aerials and microphones, in a domino-like fashion, each publication giving rise to subsequent questions and publications.

2.4. Other forms of information services for journalists

Apart from the above-mentioned information service forms and systems for the media, we prepare multi-table and multi-disciplinary orders placed by editors. These are mostly concerns of main magazines and newspapers, and often specialised television. Such kinds of work for the benefit of the media are very burdensome, laborious and time-consuming, since it is often accompanied by time pressure. There are more and more frequent complex inquiries, of which a key element is the crisis as well as social and economic threats it involves.

A separate form of both information service and information confrontation constitutes response to publications and media complaints about statistics. It is a very demanding situation, since procedures of disclaimers and clarifications, which are stipulated by Press Law, gives media a large handicap. Receiving a denial, clarification or publication of one's own version of events is extremely difficult, and achieving such an effect is beset with deliberately posed obstacles and pitfalls. Some of the leading daily newspapers have hired highly specialised legal offices to protect their publications. From the point of view of the social media effect, a received disclaimer is of third-rate importance, because the social effect on receivers' awareness has been exerted and it is difficult to reverse it by means of a delayed and limited denial.

Some confrontations assume very extensive forms and are harmful to research conducted by the public statistics. A discussion, which started in one of the newspapers concerning the SUPER BASE allegedly prepared by CSO, may serve as an example. The cause of it was a draft of the Census Act planned for 2011. Despite many announcements, disclaimers, clarifications and protests, this tsunami of absurdity, which started in February this year, has been present up to the moment of editing this text (the beginning of April 2009).

2.5.Special prospects of work with the media in 2009

As was mentioned in the introduction, since the beginning of 2009 several conditions have occurred that will affect the media service and official statistics. These are: classification of the national economy, electronic reporting, advancement of works on information systems of the official statistics (in Polish – SISP) (within the framework of the Public Administration Computerisation Plan), the change in the way economic entities are registered (in Poland it is called the principle of 1 counter), in the preparations for the censuses of 2010–2011 and preparations for starting an educational portal and a substantial modernisation of information portal.

These new circumstances have caused and will cause a substantial change of both the subject matter of the information service for journalists and the technical and organisational form of such activities. It seems to be inevitable that an online-newspaper and information chats will start, and to a larger extent official statistics services in the field of the media service will be activated.

Currently the press staff consists of Spokesman for the President of the CSO, the Press Office and the direct resources of Information Division. In each of 16 statistical offices, there is a person trained in the field of the media service, and media training is conducted, and also efforts have multiplied to speed up publications and results, as well as to enrich the informational and analytical level of presented materials.

The specialisation of voivodeship statistical offices is of utmost importance for the future of information activity for the benefit of the media. In practice, it means that what so far has been at the disposal of the specialised units at the CSO, has now been assigned to regional offices, which from now on will combine a multi-field regional function with a one field all Polish specialisation. In the case of information services, it entails completely new work conditions, the necessity for special training and a special communication system, as well as an active participation by the spokesman and his personnel in the information and media activities of statistical offices in voivodeships. So far, the results are still promising, although they also indicate a need to create a national network in the public statistics for the benefit of the media. The system under construction at present, which has been conventionally called „Library" and should be called „Information", will combine and release information resources of the Central Statistical Library (one of the largest statistical libraries in Europe), of the Central Statistical Information Centre, Press Spokesman and Press Office as well as Information Centres in 16 voivodeship offices. The Centre of Statistical Education is included in the network. Moreover, the existing data bases and data banks, as well as those that are being developed and planned, will serve the purposes of the information system for the media.

3. Official statistics and the media – mutual relationships in the 2015 perspective

2015 is the final year for the research programme for the years 2008-2015. During this period several solutions concerning the mutual relationship between official statistics and the media will be developed, which at present appear only as preliminary drafts. However, the following basic solutions for the information service for the media will remain unchanged:

- the immediate current service, mainly by means of an electronic system; the Internet, and mail chat,
- the analytical elaboration of announcements, communiqués and information notes
- the organisation of conferences and subject-related meetings (branch-related)
- the establishment of media laboratories and workshops for statistical services
- the immediate creation of own on-line newspaper
- the full reconstruction of the information portal, and, in the long-term, creation of a separate portal for cooperation between the statistics and the media.

The relationship between the media and official statistics poses a separate problem. The attitude of journalists towards statistical information is based on an excessive conviction about their own subject-related competence and an actual lack of basic preparation for conducting basic statistical analysis. Some interpretations of statistical data performed by the media are shocking and numerous cases of "statistical handicrafts", incredibly naive attempts to conduct research and statistical observations, as well as the unimaginably simplified attempts to produce different forecasts, indicate that dialogue in this field is and will be difficult. An emotional language is often overused in assessments presented by the media, and unfortunately they deprecate information obtained from statistical surveys conducted by the official statistics. In this situation the mechanism of adulating public opinion or something that is regarded as the public opinion is taking place. An example of such a trend is a series of the media messages, which anticipated the economic crisis in Poland much earlier than its symptoms occurred. Any attempt to present actual data is in general perceived as projecting an excessively-positive image, which, according to the media, has occurred in the case of the crisis.

The dialogue between official statistics and the media has to be conducted as intensively as possible on all subjects for which clarification will be demanded by the media. The media, mostly through electronic media, are the most important transmitter of information to public opinion. They shape public opinion and the awareness of the social and economic situation developed by the system of authorities. They create the behaviour of the consumers, the attitudes of producers and they substantially influence scientific assessments of the situation in the country. In statistical terms, the media presentation of statistical data is as a laser form of information transmission, which dominates, limits and deletes other forms. No reasonable person acting in the field of statistics can exclude such a form from the cooperation with the media; however, no one can neglect the work on cooperation with the media and on shaping its behaviour in the field of statistics.

STATISTICS IN TRANSITION-new series, July 2009
Vol. 10, No. 1, pp. 163—170

THE MEASURES OF ADAPTABILITY IN POLAND'S SYSTEM OF OFFICIAL STATISTICS UNDER CRISIS

Marek Cierpiał-Wolan

ABSTRACT

This paper presents the issue of ability of the Polish official statistics information system to adapt to dynamically changing economy. The main focus was both on internal changes of the organisation and public statistics relation with environment, that is government and self-government institutions, business and the media. It has been shown that due to the economic crisis not only the demand for statistical information is higher but also the role of official statistics in the information market in its broad sense is becoming more and more significant.

Key words: information market, public statistics, crisis.

1. Basic features of the information market

In so-called real sphere of economy, all the major economic agents – either in market or non-market sectors, including public and non-governmental units – make their decisions under uncertainty, partly due to incomplete and asymmetric information. The scope of uncertainty is even greater in the abruptly changing economic environment. One may distinguish between uncertainty associated with the future in contrast to the present, on the one side, and between uncertainty of market conditions and uncertainty of behaviours, on the other. For each rationally acting agent information is crucial (as a part of the rationality) and, consequently, it becomes a subject of market transaction. Of course, there is an expectation that benefits from acquiring information exceeds the costs incurred in collecting it independently.

Crisis phenomena observed in the world economy deepen the above mentioned areas of uncertainty to a large extent, causing the need for more and more information among entities belonging to all sectors. In terms of collected information, public statistics occupies a dominant position in Poland, similarly as in other countries. Throughout nearly all the period of economic transformation, one could observe insufficient use of the Polish official statistics potential in satisfying different kinds of needs on the dynamically developing information market, little flexibility of institutions in reacting to market's needs and unsatisfactory quality of statistical analyses. At the same time, a growing competition from various kinds of agencies, associations and specialized firms was causing a slow process of marginalization of public statistics on the information market.

One of the most significant aims of public statistics is to enhance the supply of information to the general public, business and non-governmental sectors in order to compensate for the information asymmetry. In this sense, one should regard information as a public good provided by the state in its autonomous activity in socio-economic field. Nowadays, in modern societies the borders between private, public, tariff and common goods are blurred. The main reason of the above is that the social costs of producing public goods are very high. Some of the methods to reduce social costs of functioning of public institutions, which provide such goods, is commercialisation, public and private partnership, as well as privatization of the entire or just the part of activity. The decision on taking active part in information market will undoubtedly force Polish Public Statistics to make use of this form of activity. It is worth stressing that during the crisis the role of the state and public goods is becoming more significant. Consequently, the activity of public statistics on the information market in its broad sense seems to be a matter of growing interest.

2. The Polish statistical information system in the time of economic changes

In the last two years, several innovative actions in information infrastructure have been taken not only within the organization itself, but they were also aimed at ensuring integrity of the information system in the whole country. Such aspirations to entirely coordinate the system is particularly important when statistical information produced by different institution is dispersed. A natural consequence of such actions should be a common policy of dissemination for all institutions. A tool used for carrying out this policy would be an online information portal, administered by CSO as a unit coordinating the whole system. It will enable public statistics to strengthen its position on the information market not only by offering standard products, but also new ones that will match the customers' needs in rapidly changing conditions.

Expectations towards statistics concern mainly two aspects. Firstly, information should be "fast", and, secondly, statistical data should refer to possibly the lowest levels of aggregation in classification and spatial breakdown. Essential precondition for accomplishing these undertakings is to shorten the process of collecting, handling and disseminating data by public statistics, as well as using preliminary and estimated data more frequently. It should be emphasised here that in Poland the time lag between collecting data and making them available is one of the shortest in Europe. What should be always remembered, though, is that the short time lag cannot reduce any of the quality dimensions: relevance, accuracy, timeliness and punctuality, accessibility and clarity, comparability and coherence.

Public statistics should pay more attention to the quality on every stage of data collecting and processing, especially in the time of crisis. A low completeness in collecting information can distort the image of dynamically changing reality to a high degree.

3. Innovations in the Polish official statistics

Statistics and government and self-government institutions

Obviously economic crisis is the reason for a greater need for statistical data concerning socio-economic phenomena, especially in the field of short-term statistics. Therefore government and self-government institutions, both on the national and regional level, turn to public statistics for preparing more detailed data that will go far beyond the established system of disseminating information.

In response to greater demand for information from these institution, public statistics establishes and publishes on websites a uniform set of indexes for monitoring socio-economic phenomena. Representatives of statistical offices are being appointed for various kinds of teams which mainly focus on monitoring socio-economic processes.

Statistics and business

Slowdown in the economy forces also business sector to become more interested in data of statistical institutions. Firms running business, especially in tough economic conditions, require a greater number of detailed information on their competition environment and turn directly to statistical offices more frequently. On this account, it is important to strengthen potential of sections responsible for information dissemination in order to meet growing needs for information in the business sector so as to identify the needs of individual customer (*personalization*), as well as customize the products to individual user's requirements (*customization*).

What seems to be necessary is greater automation of business customer service which consist among others in creating and developing a generator (software) of information package available via Internet for entrepreneurs. This kind of software will serve for making a socio-economic diagnosis needed for specifying conditions for running activity in a given area, especially useful while taking investment decisions as well as applying for projects financed by the European Union. Advanced version, with an analytical part should be used by statisticians for more complicated orders. Another action undertaken by the Polish official statistics concerns optimization of entrepreneurs' reporting burden because firms especially on the point of bankruptcy are very sensitive to additional administrative load.

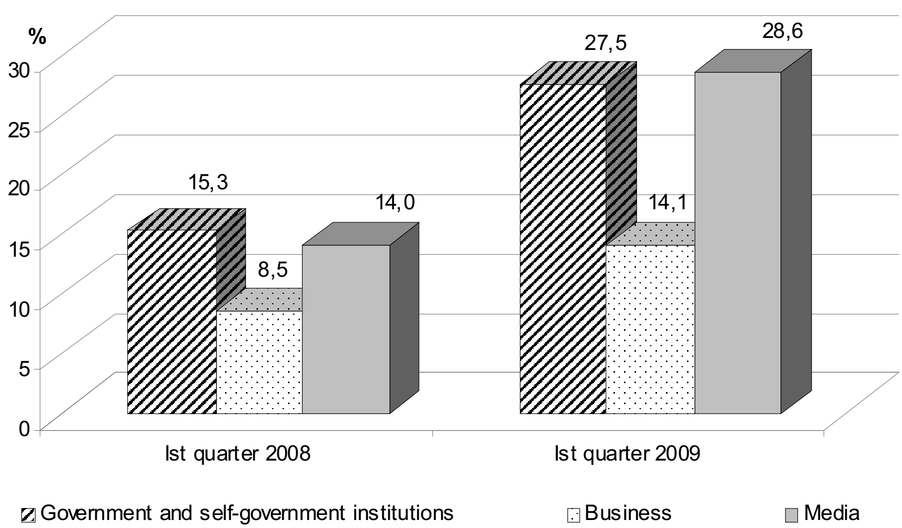
Functioning of recently launched reporting portal enables firms to receive feedback, which facilitates comparisons especially in terms of financial results for a group of enterprises with similar scope as well as for competitive environment.

Official statistics could also help entrepreneurs by creating cross-border data banks. Using registers of entities of the national economy, firms can penetrate neighbouring country market in order to launch activity or find markets abroad.

Statistics and the media

Recent years have seen a worrying phenomenon of crowding out superior quality information by inferior one. Therefore cooperation between public statistics and the media is becoming especially important. As part of the actions towards improvement of this relationship statistical offices are developing information materials well edited for the mass media, facilitating contact via the Internet, e.g. by providing a special tab for the mass media on the website. Institutions of official statistics organise also more frequent briefings and meetings with journalists aiming at systematic access to statistical information for society and proper interpretation of transferred information, especially in the time of crisis, when information "noise" is becoming higher.

Chart 1. Growth dynamics of enquires about statistical data (previous period = 100)



The chart 1 shows an increasing tendency in enquiries about statistical data from government and self-government institutions, business and the media. The number of enquiries almost doubled in the first quarter 2009 compared to first quarter 2008.

4. Intra-organization changes

For the last two years, several innovative actions have been undertaken in order to improve the functioning of the organization as well as to strengthen its official position and change its image.

One of such actions was implementation of the reporting portal. Involving all entities in on-line reporting improves data quality and shortens the time of data processing.

Another project being carried out is establishing thematic data basis, which will constitute a consistent set of information from any available sources, concerning individual socio-economic phenomena. It will ensure integrity of the country's information system and, through the system of meta-information, correct interpretation as well as the use of statistical data.

In Poland, there is a specific employment structure in public statistics – most Polish statisticians work in the regional statistical offices. The way for effective allocation of human resources was a specialization which contributed to establishing analytical and expert groups in particular research areas of statistics.

As part of improvement process there is of course inventory of information resources of statistics as well as designing new and modification of some surveys so as to adapt public statistics to changing socio-economic situation.

In the time of fluctuating economy, public statistics should include new phenomena in their survey programme. It should be noticed that crisis phenomena in countries that Poles emigrated most frequently to, increases the number of returns to our country. Since such a situation influences numerous aspects of life, they should be included into more detailed surveys and analyses. The CSO systematically publishes on websites studies concerning migration on regional and national level, based on statistical surveys and external information sources (also in cooperation with international institutions).

Table 1. Economic aspects of migration in Poland

Observed effects	Short and long-term results of migration (possible scenarios of events)
Labour market • decrease in unemployment rate, • deficit in some professions (health care,construction workers etc.) • brain drain (generally young and higher educated), • slight pay rise.	• further decrease in unemployment rate, • accelerating the process of decreasing people in working age, • increase in immigration, • higher pay rise dynamics.
Inflation • similar to average level in EU-27.	• increase in inflation (caused by increase both wages and remittances) and as a consequence increase in interest rate and possible decreasein export.
GDP • upward tendency (GDP per capita makes up 53% of average in UE-27), • increase in consumer and investment demand thanks to mainly remittances.	• deceleration of economic growth.
Remittances • steady increase.	• decrease in remittances due to different type of migrants behaviour.

The above table shows economic aspects of migration in Poland not only as observed effects but also as short and long term results migration divided into labour market, inflation, GDP and remittances. Econometrics model are being created on the basis of these specified issues. Obviously these models are used to create difference forecasts.

An excellent example can be also depreciation of national currency, which has caused higher economic activity in the cross-border areas. In general, functioning of a consistent survey system for cross-border areas will enable information on local, regional, nationwide and international level to be used extremely effectively. Effectiveness of such system requires support from standardized sources of information (official registers, bank registers, traffic automatic measure, etc.) and creation of projects that will concentrate on processes observed not only on borders but also in the border areas. Recently we have launched surveys on external and internal borders of the European Union (survey of goods and services turnover in border traffic on Polish–Ukrainian border, pilot survey of the volume of vehicle traffic on Schoengen borders). Results of the surveys can be an important supplement to statistical information on total turnover of foreign trade, better assessment of National Account and Balance of Payments, migration statistics, tourism and transport and government and self-government policies.

In order to create a uniform survey system one should establish cooperation between countries in setting subjects of survey covering the most current socio-economic issues occurring in border areas of course taking into consideration its unique character. Thus, it is significant to identify the most important endogenous and exogenous factors effecting economy and regional development in order to include them in the process of designing and carrying out surveys, as well as disseminating information. Undoubtedly standardization of methods as well as conducting joint surveys on these areas are also important.

In the time of increased need for information from statistical offices, they are expected not only to provide information, but also to analyse and interpret them in a detailed way. Therefore, selection and using of survey methods and tools for data analysis that require more than usual expert knowledge is particularly important. The example quoted here can be the use of different seasonal adjustment methods (e.g. Tramo-Seats and Arima-X12) in national accounts, which, in case of lack of thorough analysis, can show contradictory results in the time of great changes – a slight increase or slight decrease in GDP. It can greatly undermine trust in public statistics.

What appears to be necessary is to institutionally strengthen this organization in a way that will enable it to use and develop existing methods for more advanced analyses of socio-economic processes.

A key issue is constant improvement of communication with receivers of information (FAQ, Newsletter, RSS; user satisfaction questionnaire, etc.) by means of developing IT infrastructure, especially in terms of visualization of data, and more particularly in terms of users' interfaces in data banks available on websites.

Conclusion

The ongoing economic crisis world-wide, including Poland, the demand for reliable and timely information are increasing. Monitoring of socio-economic processes has become fundamentally important to government and self-government institutions, business sector and public opinion at all levels (the micro, mezo and macroeconomic). Thus, owing to the current economic crisis, the role of official statistics in the information market is becoming more and more significant. While as a 'public good' official statistics becomes better suited also to the information market's demand during the crisis, its challenges put the system of official statistics under the pressure of modernization and adaptation to new environment, including numerous innovations aimed at improving quality and availability of the produced data and information.

REFERENCES

CIERPIAŁ-WOLAN M., Asymetria informacji, efektywność i e-ekonomia, Wiadomości Statystyczne nr 2, Warszawa 2002,

CIERPIAŁ-WOLAN M., Economics aspects of migration in Poland, Forum Statisticum Slovacum 3/2007, Bratislava 2007, s.50–57.

CIERPIAŁ-WOLAN M., Cross-border surveys – some methodological aspects, Statistic in Transition new series, vol.9, No. 2, Warsaw 2008, s. 361–366.

OLEŃSKI J., Infrastruktura informacyjna państwa w globalnej gospodarce, Nowy Dziennik sp. z o.o. i Uniwersytet Warszawski, Warszawa 2006.

STATISTICS IN TRANSITION-new series, July 2009
Vol. 10, No. 1, pp. 171—173

REPORT

SCORUS CONFERENCE“ INCOME AND LIFE CONDITIONS”
BERLIN, 6-8 APRIL 2009

From 6th to 8th April 2009 the annual conference SCORUS (Standing Committee on Regional and Urban Statistics) took place in Berlin. The meetings in Berlin are entitled „Youth Assistance in Big Cities and Statistics”. „Income and Life Conditions” was the main subject of the conference in April. The standard of living, poverty, differences in the income levels were crucial and most strongly discussed issues of the social policy amongst both the countries of the European Community and all over the world. Thus, there was a need for a wider discussion amongst the scientists invited to this year's conference.

Guest from several countries such as China (Hong Kong), Japan, the USA, Canada, The Great Britain, Finland, Sweden, France, Switzerland, Portugal, the Czech Republic, Hungary and Poland participated in the meeting in Bildungs - und Begengungscentrum „ Clara Saulberg”. During the three days of the conference nearly 20 papers were presented.

The meeting was opened by its organizer – prof. dr Eckhart Elsner. Each speaker was given 20 minutes for both presenting the paper and discussion. Although the subject of the conference was specific, the problems discussed in the papers were of various character. Fundamentally, they could be divided into four main thematic branches: tax issues, the comparison of situation of young people in different countries, living conditions, budgets and quality of life and Urban Audit. The Urban Audit provides European Urban statistics for 258 cities across 27 European countries. It contains almost 300 statistical indicators presenting information on matters such as demography, society, economy, environment, transport, the information society and leisure.

Numerous discussions, which were eagerly conducted by the participants, gave the possibility of information and experience exchange. Moreover, they enriched the thematic scope of the conference by touching additional issues and expanding the subjects of the papers.

As in previous years, the organizer provided various attractions. The participants were given the opportunity to visit Reichstag and Bioshare in

Potshdam where they could see many exotic plants. They could also visit Villa Thiede built as summer residence of Johan Hampspohn in 1906.

The concert of the Verdi-Choir was a spectacular ending of the last day of the conference. It presented the pieces of Schubert, Schumann, Brahms and Mendelssohn.

It is worth noting that the atmosphere and effectiveness of these meetings are influenced by the fact that many participants of the conference in Berlin are persons who, for years, have been taking part in it at the invitation of prof. Dr Eckhart Elsner.

The titles of the papers and their authors:

1. It has been 20 years now since the Berlin Wall fell. Again of the traditional views on the history of Berlin – prof.dr Eckhart Elsner.
2. The effect of taxation and social benefits on income distribution - Kam – To Dominic Savio Leung, China.
3. Youth and housing conditions in France – prof. dr Maurice Blanc, France.
4. Portuguese Urban Areas Typology – Alexandra Isabel Marques Rodrigues Correia, Portugal.
5. The situation of the young people in Berlin, Detroit and Toronto – an attempt of the comparison – prof. Eileen Trzciński, Canada.
6. The comparison of life condition indicators of young people from selected European countries – Dorota Olszewska and Anna Żakowska, Poland.
7. Living conditions and budget of young people in Switzerland – Huguette McCluskey, Switzerland.
8. Socio-economic situation of disabled people in Poland – Sabina Augustyn and dr Paweł Ulman, Poland.
9. German statistics on income – research data centres as providers of macrodata – Dagmar Pattloch, Germany.
10. Income, life conditions and poverty in Helsinki – Markku Lankinnen, Finland.
11. The economic-institutional perspective on cultural participation of young people in Poland – Michał Chlebicki, Poland.
12. Income and taxes duties of families in Wrocław – dr Marek Kośny and Edyta Mazurek, Poland.
13. Modelling sovereign credit ratings – Anna Król and Klaudia Przybysz, Poland.
14. Income and living conditions in the periphery – epistological issues, Dereck Bond, UK.
15. Living conditions of households in Prague and some of subjective remarks on the standard of living – Jana Podhorska and Jitka Weissova, Check Republic.

16. Meassuring the quality of life in Budapest and other European cities – Norbert Bakos, Hungary.
17. Small area statistics about children's living conditions in Stockholm County and some about the rest of Sweden – Ulla Moberg, Sweden.
18. Income, consumption, housing and stocks of durables of young families in Poland – prof. dr Izabella Kudrycka and Małgorzata Radziukiewicz, Poland.
19. Changing shape of income distribution in Japan – prof. Tsunomu Tanaka, Japan.

Prepared by
Klaudia Przybysz

STATISTICS IN TRANSITION-new series, July 2009
Vol. 10, No. 1, pp. 175—177

REPORT

18TH DIDACTICAL CONFERENCE ON "QUANTITATIVE METHODS IN GLOBAL ECONOMY"

ŁÓDŹ, 1–2 JUNE 2009

The 18th Didactical Conference "Quantitative Methods in Global Economy" was organised by the Institute of Statistics and Demography and Institute of Econometrics of the University of Łódź. The mail goal of the conference was to exchange ideas and experiences concerning the teaching of quantitative methods at the academic level in the conditions of global economy.

There were over 100 participants from different academic and research centres in Poland as well as from different local government offices in Łódź. During the conference 13 papers were presented. In addition, a special panel discussion "Crucial Problems of Contemporary Demography" and a presentation of applications of STATISTICA package were organized.

The conference was opened by Prof. Czesław Domański and Prof. Mariusz Plich – Members of Organising Committee. The opening speeches were also given by Prof. Włodzimierz Nykiel – Rector of the University of Łódź, Prof. Jan B. Gajda – Dean of the Faculty of Economics and Sociology UŁ.

The first session (chair: Prof. Jan Zawadzki) started with the paper by Czesław Domański, entitled: "Educating Assistant Lectures and Ph.D. Students to Teach Quantitative Subjects". The author presented the importance of educating junior members of academic staff in conditions of social demand for statistical methods and their applications. Jan Kordos presented a paper on "Practical Experience of Teaching Mathematical Statistics and Statistical Inference at Economic Universities". The discussion was focused on difficulties the author has encountered while teaching certain topics in mathematical statistics, particularly law of large numbers and central limit theorem. In addition some terminology differences in Polish handbooks on mathematical statistics were indicated by the author.

The second session (chair: Prof. Czesław Domański) opened with the presentation by Tomasz Jurkiewicz, entitled "On Determining Quantiles in Frequency Interval Distributions". The author discussed the problem of using interpolation formulae in estimating quantiles with the assumption that the distribution of the variable is uniformly distributed in each interval is not fulfilled. New formulae depending on the frequencies of neighbouring intervals was suggested, and results of simulation experiment of comparing the efficiency of both kinds of interpolation formulae were presented.

The next speakers, Tomasz Jurkiewicz and Arkadiusz Kozłowski, presented a paper entitled "On Determining the Mode of a Continuous Variable in Raw Data". The authors considered seven different methods of estimation the mode presented in literature. The evaluation of these procedures (involving the efficiency of estimation and simplicity of computation) was made using simulation experiments.

The next presentation, given by Tomasz Jurkiewicz and Teresa Plenikowska-Ślusarz, was entitled "On Estimates of the Model of Changes over Time with Periodical Component". It was focused on the smoothing methods in time series analysis, particularly on the analytical method of coefficients. Two modification proposals of the method of coefficients were presented with the comparisons of the results obtained by the classical and modified methods which were carried out on the basis of simulation experiment and some empirical examples.

The next speaker, Barbara Olbrych, in the lecture on "Organization of Direct Interview with Survey Questionnaire to Conduct Research into Quality of Services", presented the remarks addressed at conducting a research into quality of services provided by the HELIOS Film Centre in Radom. The author presented the construction of the quality evaluation map and presented calculated satisfaction indicators, as well as, using the survey questionnaire and the IPA technique to research quality of services.

The last speech in this session, entitled "On the Use of Indicators of Dynamics of Financial Processes to Indicate the Symptom of Crisis on the Example of Zelmer S.A. (2007–2008)", was given by Emilia Sieńko. The author presented the results of the analysis of chosen indicators of dynamics of financial processes with the recommendation of a proper choice of the indicators which could be used in the analysis of symptom of crisis in Zelmer S.A.

The third session (chair: Prof. Krystyna Pruska) started with the paper by Czesław Domański, Jacek Białek, Katarzyna Bolonek-Lasoń and Artur Mikulec entitled "The Analysis of the Evaluation of the Teaching Process Based on the International Rankings of Universities". The authors presented statistical methods for evaluation of the teaching process using preference aggregation methods, the Dodgson ranking and cluster analysis in particular. These methods were used for two international rankings: Academic Ranking of World Universities and The QS World Universities, indicating a rank of Polish universities changing in time.

The next speaker, Grzegorz Kończak, in the lecture on "Internet Based Tests and Quizzes Systems – a Selective Review" described tools for producing tests and quizzes for the inter- or intranet. The author presented also the system of statistical quizzes prepared in the University of Economics in Katowice.

The next paper on "The Original Computer Program STAT_STUD as a Tool for Supporting a Teaching of Statistics" was presented by Jacek Białek. In the speech original computer program STAT_STUD was presented as free, very functional and easy to use. Its modules and merits were also shown.

Iwona Bąk and Katarzyna Wawrzyniak gave a lecture on "The Estimate of the Knowledge Usefulness at Quantitative Subjects in Professional Work in the Light of the Survey Research Made on the Extra-Mural Studies". The authors discussed results of survey research which was made among students of the extra-mural economic studies at Faculty of Economics of West Pomeranian University of Technology in Szczecin, the year 2008/2009.

The last presentation, given by Janusz Kupczun, was devoted to "New Conceptions for Introducing of Definite Integrals in University Lectures of Mathematics". A brief discussion of some recent ideas (fundamental concepts of integrals), that could be useful for intuitive and more modern teaching of calculus (i.e. mathematical analysis).

An important point of the conference programme was a special panel discussion "Crucial Problems of Contemporary Demography" organised in honour of Prof. Jerzy Tadeusz Kowaleski on the occasion of his 70th birthday and 40th anniversary of academic work. The participants of the discussion were the following:

Prof. Elżbieta Gołata, Prof. Janina Jóźwiak (moderator), Prof. Irena Kotowska, Prof. Jerzy Tadeusz Kowaleski, Prof. Jolanta Kurkiewicz, Prof. Ireneusz Kuropka, Prof. Jan Paradysz, Prof. Agnieszka Rossa, Prof. Biruta Skrętowicz, Dr Piotr Szukalski.

After the third session, the conference was closed by Prof. Czesław Domański. Prof. Mariusz Plich announced the next, 19th Didactical Conference "Teaching Quantitative Subjects and the Needs of Labour Market". The conference will take place on 7-8 June 2010. Persons wishing to take part in the conference are kindly requested to send information using the e-mail address dydaktyczna@uni.lodz.pl, or the following address:

dr Magdalena Ulrichs
Institute of Econometrics University of Łódź
Rewolucji 1905r. nr 41, 90-214 Łódź
(with XIX Didactical Conference)
tel. 042 6355525, 042 6355248, fax 042 6355248

Prepared by
Aleksandra Baszczyńska,
University of Łódź, Poland