STATISTICS IN TRANSITION new series, September 2019
Vol.20, No. 3, pp. 57–79, DOI 10.21307/stattrans-2019-024
Submitted – 21.02.2018; Paper ready for publication – 28.05.2019

GENERALIZED PARETO DISTRIBUTION BASED ON GENERALIZED ORDER STATISTICS AND ASSOCIATED INFERENCE

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ABSTRACT

In this paper, we have considered the generalized Pareto distribution. Various structural properties of the distribution are derived including (quantile function, explicit expressions for moments, mean deviation, Bonferroni and Lorenz curves and Renyi entropy). We have provided simple explicit expressions and recurrence relations for single and product moments of generalized order statistics from the generalized Pareto distribution. The method of maximum likelihood is adopted for estimating the model parameters. For different parameter settings and sample sizes, the simulation studies are performed and compared to the performance of the generalized Pareto distribution.

Key words: generalized order statistics, generalized Pareto distribution, single and product moment, recurrence relations, characterization and maximum likelihood estimation.

1. Introduction

The Pareto distribution has been introduced as a model for the distribution of incomes. It is also used as a model for losses in property and casualty insurance. The Pareto distribution has a heavy right tail behaviour, making it appropriate for including large events in applications such as excess-of-loss pricing[see Arnold (2008) and Verma and Betti (2006)]. The Pareto distribution has probability density function

$$f(x;\alpha,\beta)=\frac{\alpha\beta^{\alpha}}{(x+\beta)^{\alpha+1}};\ \ x>0,\ \alpha,\beta>0,$$

and the corresponding cumulative distribution function is

$$F(x;\alpha,\beta)=1-\left(\frac{\beta}{x+\beta}\right)^{\alpha};\ \ x>0,\ \alpha,\beta>0,$$

where β is a scale parameter and α is the shape parameter. Consider the transformation $Y=X+\beta$ to get another form of the Pareto distribution

$$f(y;\alpha,\beta)=\frac{\alpha\beta^{\alpha}}{y^{\alpha+1}};\ \ \beta\leq y<\infty,\ \alpha,\beta>0.$$

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This study uses the concept of generalized order statistics (GOS), introduced by Kamps (1995), that enables a common approach to several models of ordered random variables, such as ordinary order statistics, record values, progressively Type II censoring order statistics, Pfeifer records and sequential order statistics. The use of such a concept has been steadily growing over the years. Well-known properties of order statistics, progressively censored order statistics and record values can be subsumed, generalized and integrated within the concept of GOS. This concept can be effectively applied, e.g., in reliability theory. The statistical properties and the estimation problems based on generalized order statistics for some lifetime distributions has been studied by several researchers. For instance, Aboeleneen (2010) discussed Bayesian and non-Bayesian estimation methods based on GOS for Weibull distribution. Estimates of the unknown parameters and confidence intervals from progressively type II censoring and record values are obtained. Burkschat (2010) derived the best linear unbiased and best equivariant estimators in location and scale families of GOS from generalized Pareto distribution. Safi and Ahmed (2013) obtained the estimates of the unknown parameters of the Kumaraswamy distribution based on GOS using maximum likelihood method. Recently, Wu et al. (2014) obtained maximum likelihood estimator (MLE) of lifetime performance index for the Burr XII distribution with progressively type II right censored sample and Kim and Han (2014) obtained Bayesian estimators and highest posterior density credible intervals for the scale parameter of Rayleigh distribution based GOS. Also, they derived the Bayesian predictive estimator and the highest posterior density predictive interval for independent future observations. Recently, Kumar and Goyal (2019a, 2019b) obtained the relations for single and product moments of order statistics from power Lindley distribution and generalized lindley distribution respectively. Kumar (2015a, 2015b) and Kumar and Dey (2017a) Kumar and Jain (2018) obtained the relations for moments and moment generating function of type-II exponentiated log-logistic, extended generalized half logistic, extended exponential and power generalized Weibull distribution based on GOS respectively. Recently, Kumar et al. (2017) and Kumar and Dey (2017b) established the relations for order statistics from extended exponential and power generalized Weibull distribution and the reference therein.

The motivation of the paper is twofold: first, to derive the mathematical and statistical properties of this distribution as well as explicit expressions for single and product moments based on GOS of generalized Pareto distribution, and second, to estimate the parameters of the model using maximum likelihood method for different sample sizes and different parameter values for the generalized Pareto distribution, which we think would be of deep interest to applied statisticians.

The remaining of the article is organized as follows. In Section 2, we derive the expressions for survival function, hazard rate function, complete moments, conditional moments, mean deviation, Bonferroni and Lorenz curves, Renyi entropy and quantile function. In Section 3 we derive relations for single and product moments of GOS from generalized Pareto distribution. The obtained relations were used to compute first for moments, variances, skewness and kurtosis of order statistics and record values. We have also derived the characterization of this distribution by using conditional moments of GOS in Section 4. In Section 5, we derive maximum likelihood estimation of the generalized Pareto distribution. In Section 6, simulations are performed for different sample sizes. Section 7 ends with concluding remarks.

2. Generalized Pareto distribution

The generalized Pareto (GP)distribution was proposed by Pickands (1975). Now it is widely used in analysis of extreme events in the modelling of large insurance claims, and to describe the annual maximum flood at river gauging station.
A random variable X has the GP Distribution with two parameters α and β if it has probability density function (pdf) given by

$$f(x;\alpha,\beta)=\frac{\alpha}{(\beta x+\alpha)^2}\left(\frac{\alpha}{\beta x+\alpha}\right)^{\frac{1}{\beta}-1},\ \ x>0,\ \alpha,\beta>0 \tag{1}$$

and the corresponding cumulative distribution function (cdf) is

$$F(x;\alpha,\beta)=1-\left(\frac{\alpha}{\beta x+\alpha}\right)^{\frac{1}{\beta}},\ \ x>0,\ \alpha,\beta>0 \tag{2}$$

The hazard rate function

$$h(x;\alpha,\beta)=(\beta x+\alpha)^{-1},\ \ x>0,\ \alpha,\beta>0$$

and the survival function

$$S(x;\alpha,\beta)=\left(\frac{\alpha}{\beta x+\alpha}\right)^{\frac{1}{\beta}},\ \ x>0,\ \alpha,\beta>0.$$

Note that for GP Distribution defined in (1)

$$\bar{F}(x)=(\beta x+\alpha)f(x). \tag{3}$$

For $\beta>0$, the GP Distribution is known as Pareto type II or Lomax distribution. For $\beta=-1$, GP Distribution reduces uniform distribution on $(0,\alpha)$. As $\beta\to 0$, GP Distribution tends to exponential distribution with scale parameter α. It is well known that the GP Distribution for $\beta>0$, provides reasonably good fit to distributions of income and property values. For more details and some applications of this distribution one may refer to Pickands (1975) and Arnold (1983). Plots of the pdf (Figure 1), hazard function (Figure 2) and survival function (Figure 3), respectively for GP Distribution when $\alpha=1,2,3$ and $\beta=1,2,3$.

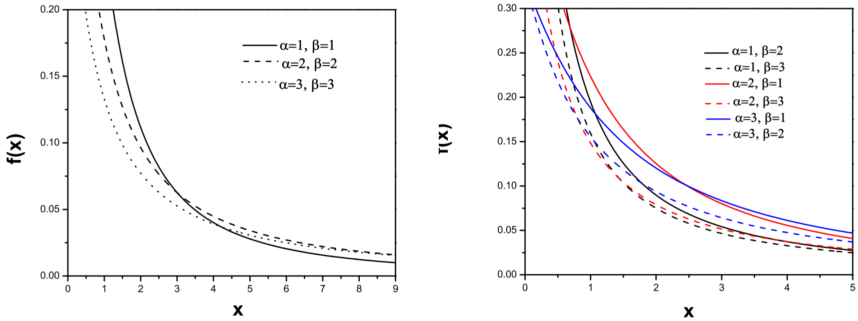

Figure 1: Probability density function of GP Distribution

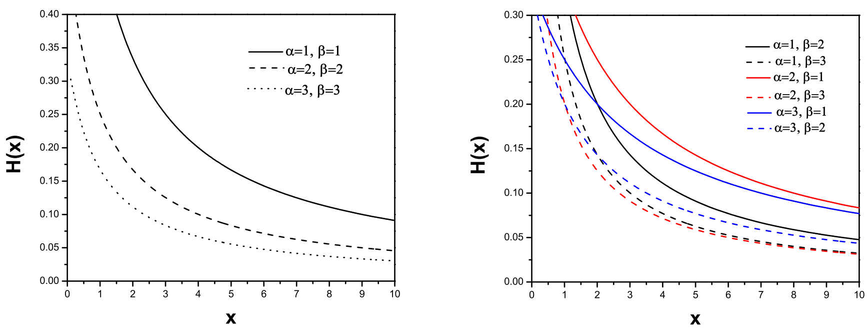

Figure 2: Hazard function of GP Distribution

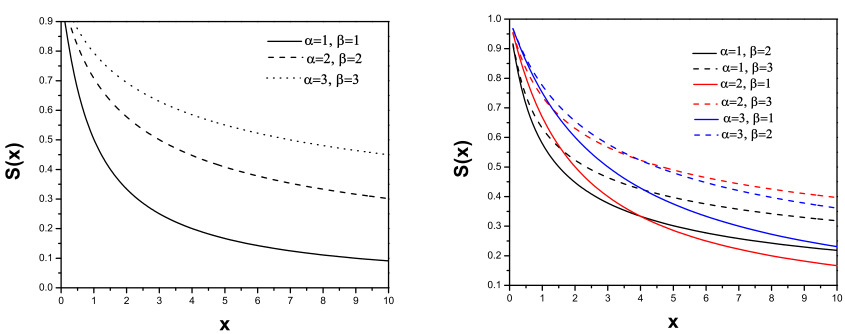

Figure 3: Survival function of GP Distribution

2.1. Quantile function

Let $x_p = Q(p) = F^{-1}(p)$, for $0 < p < 1$ denote the quantile function of the GP Distribution. then

$$x_p = \frac{\alpha[(1-p)^{-\beta} - 1]}{\beta}. \tag{4}$$

In particular, the first three quantiles, Q_1, Q_2 and Q_3, can be obtained by setting $p = 0.25$, $p = 0.5$ and $= 0.75$ in equation (4) respectively.

The effects of the parameters α and β on the skewness and kurtosis can be considered based on quantile measures. The Bowley skewness (Kenney and Keeping 1962) is one of the earliest skewness measures defined by

$$B = \frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)}.$$

Since only the middle two quartiles are considered and the outer two quartiles are ignored, this adds robustness to the measure. The Moors kurtosis (Moors 1988) is defined as

$$M = \frac{Q(3/8) - Q(1/8) + Q(7/8) - Q(5/8)}{Q(6/8) - Q(2/8)}.$$

Clearly, $M > 0$ and there is good concordance with the classical kurtosis measures for some distributions. These measures are less sensitive to outliers and they exist even for distributions without moments. For the standard normal distribution, these measures are 0 (Bowley) and 1.2331 (Moors).

2.2. Moments

Let X be a random variable having the GP Distribution. It is easy to obtain the nth moment of X as the following form

$$\begin{aligned} E(X^k) &= \int_0^\infty x^n f(x)dx = \int_0^\infty x^n \frac{\alpha}{(\beta x+\alpha)^2}\left(\frac{\alpha}{\beta x+\alpha}\right)^{\frac{1}{\beta}-1} dx \\ &= \left(\frac{\alpha}{\beta}\right)^k \sum_{p=0}^{\infty} \frac{(-1)^p \Gamma(k+1)}{p!\Gamma(k+1-p)[\beta(p-k)+1]}. \end{aligned} \tag{5}$$

The variance, skewness and kurtosis of X can be obtained using the relationship

$$Var(X) = E(X^2) - [E(X)]^2$$

$$Skewness(X) = E[X-E(X)]^3/[Var(X)]^{3/2}$$

and

$$Kurtosis(X) = E[X-E(x)]^4/[Var(X)]^2.$$

The variations of $E(X)$, $Var(X)$, $Skewness(X)$ and $Kurtosis(X)$ versus α and β are illustrated in table 1. It appears that $E(X)$, $Var(X)$, $Skewness(X)$ and $Kurtosis(X)$ are increasing function of α for every fixed β. It appears also that the $E(X)$ is greater than its $Var(X)$ for every fixed β.

2.3. Conditional moments

The conditional moments of the GP Distribution, is given by

$$\begin{aligned} E(X^k|X>x) &= \alpha\int_x^{\infty}\frac{t^k}{(\beta t+\alpha)^2}\left(\frac{\alpha}{\beta t+\alpha}\right)^{\frac{1}{\beta}-1}dt \\ &= \left(\frac{\alpha}{\beta}\right)^k\sum_{p=0}^{\infty}\frac{(-1)^p\Gamma(k+1)}{p!\Gamma(k+1-p)[\beta(p-k)+1]}\left(\frac{\alpha}{\beta x+\alpha}\right)^{p-k+\frac{1}{\beta}}. \end{aligned}$$

The mean residual lifetime function is $E(X|X>x)-x$.

Table 1: Mean, variance, skewness, kurtosis and coefficient of variation for $\beta=5$ and some values of α

α	Mean	Variance	Skewness	Kurtosis	CV
1	0.025221	0.002242	2.266105	3.254921	4.444709
2	0.050441	0.008969	2.272871	3.269397	8.890585
3	0.075662	0.020179	2.273945	3.272412	13.33496
4	0.100882	0.035875	2.274003	3.272299	17.78067
5	0.126103	0.056055	2.273847	3.272586	22.22588
6	0.151323	0.080719	2.273877	3.272552	26.67109
7	0.176544	0.109867	2.273915	3.272620	31.11604
8	0.201764	0.143500	2.273924	3.272615	35.56135
9	0.226985	0.181618	2.273875	3.272584	40.00661
10	0.252206	0.224219	2.273877	3.272596	44.45156

2.4. Mean deviations

The mean deviations about the mean and the median are used to measure the dispersion and the spread in a population from the centre. The mean deviations about the mean $\mu=E(X)$ and about the median M can be calculated as

$$D(\mu)=E|x-\mu|=\int_0^{\infty}|x-\mu|f(x)dx$$

and

$$D(m)=E|x-m|=\int_0^{\infty}|x-m|f(x)dx,$$

respectively. The measures, we obtain $D(\mu)$ and $D(m)$, can be calculated using the following relationships:

$$\begin{aligned} D(\mu) &= \int_0^{\mu}(\mu-x)f(x)dx+\int_{\mu}^{\infty}(x-\mu)f(x)dx \\ &= \mu F(\mu)-\int_0^{\mu}xf(x)dx-\mu[1-F(\mu)]+\int_{\mu}^{\infty}xf(x)dx \\ &= 2\mu F(\mu)-2\mu+2\int_{\mu}^{\infty}xf(x)dx \end{aligned}$$

and

$$\begin{aligned} D(m) &= \int_0^{m}(m-x)f(x)dx+\int_{m}^{\infty}(x-m)f(x)dx \\ &= mF(m)-\int_0^{m}xf(x)dx-m[1-F(m)]+\int_{m}^{\infty}xf(x)dx \\ &= 2mF(m)-m-\mu+2\int_{m}^{\infty}xf(x)dx. \end{aligned}$$

Consider

$$I=\int_{\mu}^{\infty}xf(x)dx. \tag{6}$$

Using the substitution $t=[\bar{F}(x)]^{\beta}$ in (6), we obtain

$$\int_{\mu}^{\infty}xf(x)dx=\frac{\alpha}{\beta(1-\beta)}\left(\frac{\alpha}{\beta\mu+\alpha}\right)^{1/\beta}\left[\left(1+\frac{\beta\mu}{\alpha}\right)+\beta-1\right]$$

and

$$\int_{m}^{\infty}xf(x)dx=\frac{\alpha}{\beta(1-\beta)}\left(\frac{\alpha}{\beta m+\alpha}\right)^{1/\beta}\left[\left(1+\frac{\beta m}{\alpha}\right)+\beta-1\right],$$

so it follows that

$$D(\mu)=2\mu F(\mu)-2\mu+\frac{2\alpha}{\beta(1-\beta)}\left(\frac{\alpha}{\beta\mu+\alpha}\right)^{1/\beta}\left[\left(1+\frac{\beta\mu}{\alpha}\right)+\beta-1\right],$$

and

$$D(m)=2mF(m)-m-\mu+\frac{2\alpha}{\beta(1-\beta)}\left(\frac{\alpha}{\beta m+\alpha}\right)^{1/\beta}\left[\left(1+\frac{\beta m}{\alpha}\right)+\beta-1\right].$$

2.5. Bonferroni and Lorenz curve

Boneferroni and Lorenz curves are proposed by Boneferroni (1930). These curves have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. They are defined as

$$B(p) = \frac{1}{p\mu}\int_0^q xf(x)dx \tag{7}$$

$$L(p) = \frac{1}{\mu}\int_0^q xf(x)dx, \tag{8}$$

and respectively, where $\mu = E(X)$ and $q = F^{-1}(p)$. By using (1), one can reduce (7) and (8) to

$$B(p) = \frac{\alpha}{p\mu(1-\beta)}\left[1-\beta\left(\frac{\alpha}{\beta q+\alpha}\right)^{1/\beta}\left\{\left(1+\frac{\beta q}{\alpha}\right)+\beta-1\right\}\right],$$

and

$$L(p) = \frac{\alpha}{\mu(1-\beta)}\left[1-\beta\left(\frac{\alpha}{\beta q+\alpha}\right)^{1/\beta}\left\{\left(1+\frac{\beta q}{\alpha}\right)+\beta-1\right\}\right],$$

respectively.

2.6. Renyi entropy

The entropy of a random variable X with the density function $f(x)$ is a measure of variation of the uncertainty. Renyi entropy is defined as $I_R(\rho) = (1-\rho)^{-1}log[I(\rho)]$, where $I(\rho) = \int_{\Re} f^{\rho}(x)dx$, $\rho > 0$ and $\rho \neq 1$. If a random variable X has a GP distribution, then, we have

$$\begin{aligned} I(\rho) &= \alpha^{\rho}\int_0^{\infty}\frac{1}{(\beta x+\alpha)^{2\rho}}\left(\frac{\alpha}{\beta x+\alpha}\right)^{\rho\left(\frac{1}{\beta}-1\right)}dx \\ &= \frac{1}{\beta(1+\beta)\alpha^{\rho}}, \end{aligned}$$

[see Gradshteyn and Ryzhik (2014), p-322]. Hence, the Renyi entropy reduces to

$$I_R(\rho) = -\frac{1}{1-\rho}log\beta(\beta+1)+\left(\frac{\rho}{1-\rho}\right)log\alpha.$$

3. Generalized order statistics

The concept of generalized order statistics GOS was introduced by Kamps (1995). Several models of ordered random variables such as order statistics, record values, sequential order statistics, progressive type II censored order statistics and Pfeifer's record values can be discussed as special cases of the GOS. Suppose

$X(1,n,m,k),\ldots,X(n,n,m,k)$, ($k \geq 1, m$ is a real number), are n GOS from an absolutely continuous $cdf\ F(x)$ with $pdf\ f(x)$, if their joint pdf is of the form

$$f_{X(1,n,m,k),\ldots,X(n,n,m,k)}(x_1,x_2,\ldots,x_n)$$
$$= k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^m f(x_i)\right)[1-F(x_n)]^{k-1}f(x_n) \tag{9}$$

on the cone $F^{-1}(0) \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq F^{-1}(1)$,
where $\gamma_j = k+(n-j)(m+1) > 0$ for all $j,\ 1 \leq j \leq n,\ k$ is a positive integer and $m \geq -1$.
If $m = 0$ and $k = 1$, then this model reduces to the ordinary $r-$th order statistic and (9) will be the joint pdf of n order statistics $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ from $cdf\ F(x)$. If $k = 1$ and $m = -1$, then (9) will be the joint pdf of the first n record values of the identically and independently distributed $(i.i.d.)$ random variables with $cdf\ F(x)$ and corresponding $pdf\ f(x)$. In view of (9), the marginal pdf of the $r-$th GOS, $X(r,n,m,k)$, $1 \leq r \leq n$, is

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x)), \tag{10}$$

and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \leq r < s \leq n$, $x < y$ is

$$\begin{aligned} f_{X(r,n,m,k),X(s,n,m,k)}(x,y) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m f(x)g_m^{r-1}(F(x)) \\ &\times [h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y), \end{aligned} \tag{11}$$

where

$$\bar{F}(x) = 1-F(x), \quad C_{r-1} = \prod_{i=1}^{r}\gamma_i\ ,$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -ln(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1),\ \ x \in [0,1).$$

3.1. Relations for single moments of generalized order statistics

We shall first establish explicit expressions for jth single moments of the rth generalized order statistics, $E\left(X^j(r,n,m,k)\right)$. For the GP distribution, as given in (1), the $j-$th moments of $X(r,n,m,k)$ is given as,

$$\begin{aligned} E[X^j(r,n,m,k)] &= \int_0^\infty x^j f_{X(r,n,m,k)}(x)dx \\ &= \frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx. \end{aligned} \tag{12}$$

Further, on using the binomial expansion, we can rewrite (12) as

$$\begin{aligned} E[X^j(r,n,m,k)] &= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} \\ &\times \int_0^\infty x^j[\bar{F}(x)]^{\gamma_{r-u}-1} f(x)dx. \end{aligned} \tag{13}$$

Now, letting $t=[\bar{F}(x)]^\beta$ in (13), we get

$$\begin{aligned} E[X^j(r,n,m,k)] &= \frac{C_{r-1}}{(r-1)!(m+1)^r}\left(\frac{\alpha}{\beta}\right)^j \sum_{p=0}^{j}\sum_{u=0}^{r-1}(-1)^{u+p}\binom{r-1}{u}\binom{j}{p} \\ &\times B\left(\frac{k}{m+1}+n-r+u+\frac{\beta(p-j)}{m+1},1\right), \end{aligned} \tag{14}$$

Since

$$\sum_{a=0}^{b}(-1)^a\binom{b}{a}B(a+k,c)=B(k,c+b), \tag{15}$$

where $B(a,b)$ denotes the complete beta function and defined by $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$
Therefore,

$$\begin{aligned} E[X^j(r,n,m,k)] &= \frac{C_{r-1}}{(m+1)^r}\left(\frac{\alpha}{\beta}\right)^j \sum_{p=0}^{j}(-1)^p\binom{j}{p} \\ &\times \frac{\Gamma\left(\frac{k+(n-r)(m+1)+\beta(p-j)}{m+1}\right)}{\Gamma\left(\frac{k+n(m+1)+\beta(p-j)}{m+1}\right)} \end{aligned} \tag{16}$$

$$= \left(\frac{\alpha}{\beta}\right)^j \sum_{p=0}^{j}(-1)^p\binom{j}{p}\frac{1}{\prod_{a=1}^{r}\left(1+\frac{\beta(p-j)}{\gamma_a}\right)}, \tag{17}$$

where $\Gamma(.)$ denotes the complete gamma function and defined by $\Gamma(a)=\int_0^\infty t^{a-1}e^{-t}dt$.
Special Cases
i) Putting $m=0$, $k=1$, in (16), we get moments of order statistics from GP distribution as

$$E[X^j_{r:n}]=\frac{n!}{(n-r)!}\left(\frac{\alpha}{\beta}\right)^j \sum_{p=0}^{j}(-1)^p\binom{j}{p}\frac{\Gamma(n-r+1+\beta(p-j))}{\Gamma(n+1+\beta(p-j))}. \tag{18}$$

ii) Setting $m=-1$ in (17), to get moments of $k-$th record value from GP distribution as;

$$E[X(r,n,-1,k)]=\left(\frac{\alpha}{\beta}\right)^j \sum_{p=0}^{j}(-1)^p\binom{j}{p}\frac{1}{\left(1+\frac{\beta(p-j)}{k}\right)^r}$$

for upper record values $k = 1$

$$E[X^j_{U(r)}] = \left(\frac{\alpha}{\beta}\right)^j \sum_{p=0}^{j} (-1)^p \left(\begin{array}{c} j \\ p \end{array}\right) \frac{1}{[\beta(p-j)]^r}. \tag{19}$$

A recurrence relation for single moment of GOS from cdf (2) can be obtained in the following theorem.

Theorem 1. For the distribution given in (1) and for $2 \leq r \leq n$, $n \geq 2$ $k = 1, 2, \ldots,$

$$\left(1 - \frac{j\beta}{\gamma_r}\right) E[X^j(r,n,m,k)] = E[X^j(r-1,n,m,k)] + \frac{j\alpha}{\gamma_r} E[X^{j-1}(r,n,m,k)]. \tag{20}$$

Proof. From (10), we have

$$E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx.$$

Integrating by parts and using (3) and simplifying the resultant expression we get the result given in (20).

Remark 1: Under the assumption of Theorem 1 with $m = 0$, $k = 1$, we shall deduced the recurrence relations for single moments of ordinary order statistics of the GP distribution

$$\left(1 - \frac{j\beta}{n-r+1}\right) E(X^j_{r:n}) = E(X^j_{r-1:n}) + \frac{j\alpha}{(n-r+1)} E(X^{j-1}_{r:n}).$$

Remark 2: Putting $m = -1$ in Theorem 1 we obtain the recurrence relations for single moments of k record values of the GP distribution.

$$\left(1 - \frac{j\beta}{k}\right) E(X^j_{U(r)}) = E(X^j_{U(r-1)}) + \frac{j\alpha}{k} E(X^{j-1}_{U(r)}).$$

All the tables and figures are made by using R software. The codes of the program are available from the author on request. Table 2-3 lists some numerical values for the first four moments, variances, skewness and kurtosis of order statistics and upper record values from equation (18) and (19) and using numerical integration. The parameter values are taken as $\alpha = 2$ and $\beta = 0.5$. The results in this table show a good agreement between the two methods.

Table 2: First four moments, variances, skewness and kurtosis of some order statistics from equation (18) and via numerical integration

$X_{r:50} \downarrow$	$j-th$ moments $\rightarrow$	$j=1$	$j=2$	$j=3$	$j=4$	Variance	Skewness	Kurtosis
$r=1$	Expression (18)	1.791126	6.076642	20.70228	71.72752	2.868510	0.004433	1.154258
	Numerical	1.787827	6.049550	20.56910	71.71503	2.853225	0.004332	1.226838
$r=5$	Expression (18)	1.171432	2.242526	4.737888	10.80944	0.870273	0.003931	1.879686
	Numerical	1.170642	2.239087	4.726975	10.77690	0.868684	0.003951	1.880719
$r=10$	Expression (18)	0.910562	1.276332	2.024636	3.532368	0.447209	0.012898	2.226165
	Numerical	0.910210	1.275245	2.022027	3.526351	0.446763	0.012913	2.226649
$r=15$	Expression (18)	0.774743	0.896511	1.186052	1.738996	0.296284	0.020178	2.407004
	Numerical	0.774531	0.895981	1.184989	1.736927	0.296083	0.020188	2.407285
$r=20$	Expression (18)	0.687348	0.692336	0.801766	1.033374	0.219889	0.026215	2.521945
	Numerical	0.687203	0.692022	0.801214	1.032429	0.219774	0.026225	2.522204
$r=25$	Expression (18)	0.624858	0.564523	0.588489	0.684613	0.174075	0.031388	2.603020
	Numerical	0.624750	0.564315	0.58816	0.684104	0.174002	0.031396	2.603190
$r=30$	Expression (18)	0.577222	0.476860	0.455715	0.486926	0.143675	0.035917	2.663920
	Numerical	0.577138	0.476712	0.455501	0.486622	0.143624	0.035930	2.664067
$r=35$	Expression (18)	0.539307	0.412941	0.366450	0.364092	0.122089	0.039968	2.711642
	Numerical	0.539238	0.412831	0.366301	0.363895	0.122053	0.039963	2.711900
$r=40$	Expression (18)	0.508172	0.364243	0.303026	0.282557	0.106004	0.043595	2.750593
	Numerical	0.508115	0.364157	0.302918	0.282423	0.105976	0.043604	2.750620
$r=50$	Expression (18)	0.459576	0.294896	0.220099	0.184408	0.083686	0.049953	2.810085
	Numerical	0.459534	0.294840	0.220036	0.184337	0.083669	0.049957	2.810084

Table 3: First four moments, variances, skewness and kurtosis of some upper record values from equation (19) and via numerical integration

$r \downarrow$	$j-th$ moments $\rightarrow$	$j=1$	$j=2$	$j=3$	$j=4$	Variance	Skewness	Kurtosis
1	Expression (19)	1.891126	6.376642	21.70228	73.61752	2.800284	0.004765	1.211188
	Numerical	1.835999	6.226602	21.21964	73.54503	2.85571	0.010477	1.171609
2	Expression (19)	1.328135	3.217226	8.587433	24.52173	1.453283	0.033607	1.712181
	Numerical	1.350241	3.278293	8.764432	25.05541	1.455142	0.027061	1.704100
3	Expression (19)	0.955521	1.682350	3.539371	8.351799	0.76933	0.234001	2.600790
	Numerical	0.964074	1.705065	3.597157	8.504109	0.775626	0.224602	2.575414
4	Expression (19)	0.673323	0.871408	1.451192	2.835484	0.418044	0.622106	3.895456
	Numerical	0.674463	0.878537	1.468775	2.877312	0.423637	0.610888	3.855009
5	Expression (19)	0.467551	0.448047	0.592478	0.960073	0.229443	1.174468	5.628818
	Numerical	0.465080	0.449381	0.597197	0.970923	0.233082	1.159966	5.573523
6	Expression (19)	0.321244	0.229045	0.241045	0.324331	0.125847	1.881817	7.858959
	Numerical	0.317367	0.228555	0.241977	0.326887	0.127833	1.866261	7.795781
7	Expression (19)	0.219019	0.116560	0.097785	0.109353	0.068591	2.760704	10.69797
	Numerical	0.214919	0.115722	0.097766	0.109843	0.069532	2.75115	10.64534
8	Expression (19)	0.148476	0.059104	0.039574	0.036809	0.037059	3.849072	14.31953
	Numerical	0.144723	0.058385	0.039407	0.03685	0.03744	3.856723	14.30993
9	Expression (19)	0.100229	0.029885	0.015984	0.012373	0.019839	5.200595	18.96155
	Numerical	0.097047	0.029374	0.015853	0.012345	0.019956	5.243194	19.04621
10	Expression (19)	0.067447	0.015077	0.006446	0.004154	0.010528	6.886594	24.93712
	Numerical	0.064874	0.014745	0.006367	0.004131	0.010536	6.987957	25.20555

3.2. Relations for product moments of generalized order statistics

We shall first establish explicit expressions for the product moment of ith and jth generalized order statistics, $E\left(X^{(i,j)}_{r,s,n,m,k}\right) = \mu^{(i,j)}_{r,s,n,m,k}$. For GP distribution, the product moment of $X(r,n,m,k)$ and $X(s,n,m,k)$ is given as

$$E[X^i(r,n,m,k),X^j(s,n,m,k)] = \int_0^\infty \int_x^\infty x^i x^j f_{X(r,n,m,k)X(s,n,m,k)}(x,y)dxdy.$$

On using (11) and binomial expansion, we have

$$E[X^i(r,n,m,k),X^j(s,n,m,k)] = \frac{C_{s-1}(m+1)^{2-s}}{(r-1)!(s-r-1)!}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}$$
$$\times \binom{r-1}{u}\binom{s-r-1}{v}\int_0^\infty x^i[\bar{F}(x)]^{(s-r+u-v)(m+1)-1}f(x)G(x)dx, \quad (21)$$

where

$$G(x) = \int_x^\infty x^j[\bar{F}(y)]^{\gamma_{s-v}-1}f(y)dy. \quad (22)$$

By setting $t=[\bar{F}(y)]^\beta$ in (22), we obtain

$$G(x) = \left(\frac{\alpha}{\beta}\right)^j \sum_{p=0}^{j}(-1)^p \binom{j}{p}\frac{[\bar{F}(x)]^{\gamma_{s-v}+\beta(p-j)}}{[\gamma_{s-v}+\beta(p-j)]}.$$

On substituting the above expression of $G(x)$ in (22), and simplifying the resulting equation, we get.

$$\begin{aligned} E[X^i(r,n,m,k),X^j(s,n,m,k)] &= \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^s}\left(\frac{\alpha}{\beta}\right)^{i+j}\sum_{p=0}^{j}\sum_{q=0}^{i}(-1)^{p+q} \\ &\times \binom{j}{p}\binom{i}{q}B\left(\frac{k}{m+1}+n-r+\frac{\beta(p+q-i-j)}{m+1},r\right) \\ &\times B\left(\frac{k}{m+1}+n-s+\frac{\beta(p-j)}{m+1},s-r\right), \end{aligned} \quad (23)$$

which after simplification yields

$$\begin{aligned} E[X^i(r,n,m,k),X^j(s,n,m,k)] &= \left(\frac{\alpha}{\beta}\right)^{i+j}\sum_{p=0}^{j}\sum_{q=0}^{i}(-1)^{p+q}\binom{j}{p}\binom{i}{q} \\ &\times \frac{1}{\prod_{a=1}^{r}\left(1+\frac{\beta(p+q-i-j)}{\gamma_a}\right)\prod_{b=r+1}^{s}\left(1+\frac{\beta(p-j)}{\gamma_b}\right)}. \end{aligned} \quad (24)$$

Special cases

i) Putting $m=0$, $k=1$ in (23), we shall deduced the explicit formula for product

moments of ordinary order statistics of GP distribution.
ii) Setting $m=-1$ in (24), we obtain the explicit expression for product moments of k record values of GP distribution.

Making use of (3), we can derive recurrence relations for product moments of GOS from (11).

Theorem 2. For the distribution given in (1) and for $1 \le r < s \le n,\ n \ge 2$ and $k = 1,2...$

$$\begin{aligned}\left(1-\frac{j\beta}{\gamma_s}\right)E[X^i(r,n,m,k)X^j(s,n,m,k)] &= E[X^i(r,n,m,k)X^j(s-1,n,m,k)] \\ &+ \frac{j\alpha}{\gamma_s}E[X^i(r,n,m,k)X^{j-1}(s,n,m,k)]. \quad (25)\end{aligned}$$

Proof: Using (11), we have

$$\begin{aligned}E[X^i(r,n,m,k)X^j(s,n,m,k)] &= \frac{C_{s-1}}{(r-1)!(s-r-1)!} \\ &\times \int_0^\infty x^i[\bar{F}(x)]^m f(x) g_m^{r-1}(F(x))I(x),dx \quad (26)\end{aligned}$$

where

$$I(x)=\int_x^\infty y^j[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(y)dy.$$

Solving the integral in $I(x)$ by parts and using (3) and substituting the resulting expression in (26), we get the result given in (25).
Remark 3 Under the assumption of Theorem 2 with $m=0$, $k=1$ we shall deduced the recurrence relations for product moments of order statistics of the GP distribution.
Remark 4 Putting $m=-1$ in Theorem 2 we obtain the recurrence relations for product moments of $k-$th record values from GP distribution.
Remark 5 At $j=0$ in (25), we have

$$E[X^i(r,n,m,k)] = \left(\frac{\alpha}{\beta}\right)^i \sum_{q=0}^{i}(-1)^q \left(\begin{array}{c} i \\ q \end{array}\right) \frac{1}{\prod_{a=1}^{r}\left(1+\frac{\beta(q-i)}{\gamma_a}\right)}.$$

Remark 6 At $i=0$, Theorem 2 reduces to Theorem 1.

4. Characterization

Let $X(r,n,m,k)$, $r=1,2,\ldots,n$ be GOS, then from a continuous population with cdf $F(x)$ and pdf $f(x)$, then the conditional pdf of $X(s,n,m,k)$ given $X(r,n,m,k)=x$,

$1 \leq r < s \leq n$, in view of (10) and (11), is

$$f_{X(s,n,m,k)|X(r,n,m,k)}(y|x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}} \times \frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s - 1}}{[\bar{F}(x)]^{\gamma_{r+1}}} f(y). \quad x < y \quad (27)$$

Theorem 3: Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0) = 0$ and $0 < F(x) < 1$ for all $x > 0$, then

$$E[X(s,n,m,k)|X(l,n,m,k) = x] = \frac{(\beta x + \alpha)}{\beta} \left\{ \prod_{j=1}^{s-l} \left(\frac{\gamma_{l+j}}{\gamma_{l+j} - \beta} \right) - \alpha \right\}, \quad l = r, \ r+1 \quad (28)$$

if and only if

$$F(x; \alpha, \beta) = 1 - \left(\frac{\alpha}{\beta x + \alpha} \right)^{\frac{1}{\beta}} \quad x > 0, \ \alpha, \beta > 0.$$

Proof. From (27), we have

$$E[X(s,n,m,k)|X(r,n,m,k) = x] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \int_x^\infty y \left(\frac{\bar{F}(y)}{\bar{F}(x)} \right)^{\gamma_s - 1} \times \left[1 - \left(\frac{\bar{F}(y)}{\bar{F}(x)} \right)^{m+1} \right]^{s-r-1} \frac{f(y)}{\bar{F}(x)} dy. \quad (29)$$

By setting $u = \frac{\bar{F}(y)}{\bar{F}(x)}$ from (2) in (29), we obtain

$$E[X(s,n,m,k)|X(r,n,m,k) = x] = \frac{C_{s-1}}{\beta(s-r-1)!C_{r-1}(m+1)^{s-r-1}} [(\beta x + \alpha)A_1 - \alpha A_2], \quad (30)$$

where

$$A_1 = \int_0^1 u^{\gamma_s - \beta - 1}(1 - u^{m+1})^{s-r-1} du \quad (31)$$

and

$$A_2 = \int_0^1 u^{\gamma_s - 1}(1 - u^{m+1})^{s-r-1} du. \quad (32)$$

Again by setting $t = u^{m+1}$ in (31) and (32) and substituting the values of A_1 and A_2 in (30) and simplifying the resultant expression we get the result given in (28).

To prove sufficient part, we have from (27) and (28)

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_x^\infty y[(\bar{F}(x))^{m+1}-(\bar{F}(y))^{m+1}]^{s-r-1} \times[\bar{F}(y)]^{\gamma_s-1}f(y)dy=[\bar{F}(x)]^{\gamma_{r+1}}H_r(x), \tag{33}$$

where

$$H_r(x)=\frac{(\beta x+\alpha)}{\beta}\left\{\prod_{j=1}^{s-r}\left(\frac{\gamma_{r+j}}{\gamma_{r+j}-\beta}\right)-\alpha\right\}.$$

Differentiating (33) both sides with respect to x and rearranging the terms, we get

$$-\frac{C_{s-1}[\bar{F}(x)]^m f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}}\int_x^\infty y[(\bar{F}(x))^{m+1}-(\bar{F}(y))^{m+1}]^{s-r-2} \times[\bar{F}(y)]^{\gamma_s-1}f(y)dy=H_r'(x)[\bar{F}(x)]^{\gamma_{r+1}}-\gamma_{r+1}H_r(x)[\bar{F}(x)]^{\gamma_{r+1}-1}f(x)$$

Therefore,

$$\frac{f(x)}{\bar{F}(x)}=-\frac{H_r'(x)}{\gamma_{r+1}[H_{r+1}(x)-H_r(x)]}=\frac{1}{(\beta x+\alpha)},$$

which proves that

$$F(x;\alpha,\beta)=1-\left(\frac{\alpha}{\beta x+\alpha}\right)^{\frac{1}{\beta}} \quad x>0,\ \alpha,\beta>0.$$

5. Estimation of model parameters

In this section we discuss the process of obtaining the maximum likelihood estimators of the parameters α and β. Let $X_1,X_2,\ldots,X_n$ be random sample with observed values $x_1,x_2,\ldots,x_n$ from GP distribution. Let $\Theta=(\alpha,\beta)$ be the parameter vector. The likelihood function based on the random sample of size n is obtained from

$$L(\alpha,\beta|x)=\alpha^{n/\beta}\prod_{i=1}^{n}(\beta x_i+\alpha), \tag{34}$$

The maximum likelihood estimates are the values of α and β that maximize this likelihood function.

5.1. Maximum likelihood estimation

The log likelihood function $l(\alpha,\beta|x) = logL(\alpha,\beta|x)$, dropping terms that do not involve α and β, is

$$l(\alpha,\beta|x) = \frac{n}{\beta}log\alpha - \left(1+\frac{1}{\beta}\right)\sum_{i=1}^{n} log(\beta x_i+\alpha). \tag{35}$$

We assume that the parameters α and β are unknown. To obtain the normal equations for the unknown parameters, we differentiate (35) partially with respect to α and β and equate to zero. The resulting equations are

$$0 = \frac{\partial l(\alpha,\beta|x)}{\partial\alpha} = \frac{n}{\alpha\beta} - \left(1+\frac{1}{\beta}\right)\sum_{i=1}^{n}\frac{1}{(\beta x_i+\alpha)}, \tag{36}$$

and

$$0 = \frac{\partial l(\alpha,\beta|x)}{\partial\beta} = -\frac{n}{\beta^2}log\alpha + \frac{1}{\beta^2}\sum_{i=1}^{n} log(\beta x_i+\alpha) - \left(1+\frac{1}{\beta}\right)\sum_{i=1}^{n}\frac{x_i}{(\beta x_i+\alpha)}. \tag{37}$$

The solutions of the above equations are the maximum likelihood estimators of the GP distribution parameters α and β, denoted as $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$, respectively. As the equations expressed in (36) and (37) cannot be solved analytically, one must use a numerical procedure to solve them.

5.2. Approximate confidence intervals

In this section, we present the asymptotic confidence intervals for the parameters of the GP distribution. Since the $MLEs$ of the unknown parameters α and β cannot be derived in closed form, it is not easy to derive the exact distributions of the $MLEs$. Hence, we cannot obtain exact confidence intervals for the parameters. We must use the large sample approximation. It is known that the asymptotic distribution of the $MLEs$ is $[\sqrt{n}(\hat{\alpha}_{MLE}-\alpha), \sqrt{n}(\hat{\beta}_{MLE}-\beta)] \to N_2(0, I^{-1}(\Theta))$, we can refer to Lawless (1982), where $I^{-1}(\Theta)$, the inverse of the observed information matrix of the unknown parameters $\Theta = (\alpha,\beta)$, is

$$I^{-1}(\Theta) = \begin{pmatrix} -\frac{\partial^2 l(\alpha,\beta)}{\partial^2\alpha} & -\frac{\partial^2 l(\alpha,\beta)}{\partial\alpha\partial\beta} \\ -\frac{\partial^2 l(\alpha,\beta)}{\partial\alpha\partial\beta} & -\frac{\partial^2 l(\alpha,\beta)}{\partial^2\beta} \end{pmatrix}^{-1}_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})}$$

$$= \begin{pmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\beta}) \\ Cov(\hat{\alpha},\hat{\beta}) & Var(\hat{\alpha}) \end{pmatrix}.$$

The derivatives in $I(\Theta)$ are given in

$$\frac{\partial^2 l(\alpha,\beta|x)}{\partial \alpha^2} = -\frac{n}{\alpha^2\beta} + \left(1+\frac{1}{\beta}\right)\sum_{i=1}^{n}\frac{1}{(\beta x_i+\alpha)^2}$$

$$\frac{\partial^2 l(\alpha,\beta|x)}{\partial \alpha\, \partial \beta} = -\frac{n}{\alpha\beta^2} + \frac{1}{\beta^2}\sum_{i=1}^{n}\frac{1}{(\beta x_i+\alpha)} - \left(1+\frac{1}{\beta}\right)\sum_{i=1}^{n}\frac{x_i}{(\beta x_i+\alpha)^2} = \frac{\partial^2 l(\alpha,\beta|x)}{\partial \beta\, \partial \alpha}$$

$$\begin{aligned}\frac{\partial^2 l(\alpha,\beta|x)}{\partial \beta^2} &= \frac{2n}{\beta^3}ln\alpha - \frac{2}{\beta^3}\sum_{i=1}^{n} ln(\beta x_i+\alpha) + \frac{2}{\beta^2}\sum_{i=1}^{n}\frac{x_i}{(\beta x_i+\alpha)} \\ &+ \ + \left(1+\frac{1}{\beta}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\beta x_i+\alpha}\right)^2.\end{aligned}$$

The above approach is used to derive approximate $100(1-\tau)\%$ confidence intervals of the parameters α and β of the forms

$$\hat{\alpha} \pm z_{\tau/2}\sqrt{var(\hat{\alpha})}$$

and

$$\hat{\beta} \pm z_{\tau/2}\sqrt{var(\hat{\beta})},$$

where $z_{\tau/2}$ is the upper $(\tau/2)$th percentile of the standard normal distribution.

6. Numerical Experiments and Discussion

In this section, we examine the performance of maximum likelihood estimates for the two parameter GP distribution by conducting simulation study for different sample sizes $n = 20, 30, 50, 100, 150$. We simulate 1000 samples with four different sets of parameters. The results are presented in Table 4, which shows the averages of $MLEs[Av(\hat{\alpha},\hat{\beta})]$ together with the 95% confidence intervalsfor parameters of GP distribution $[C(\alpha,\beta)]$ and their variances, $[Var(\hat{\alpha}),Var(\hat{\beta})]$. These results suggest that ML estimates performed adequately. The variances of $MLEs$ decrease when the sample size n increases.

The following observations can be drawn from the Tables 4

1. All the estimators show the property of consistency i.e., the $MLEs$ decreases as sample size increases.

2. The variances of $MLEs$ decrease when n increases.

Table 4: Mean of the $MLEs$, their variances and confidence interval

n	(α,β)	$Av(\hat{\alpha},\hat{\beta})$	$C(\alpha,\beta)$	$Var(\hat{\alpha})$	$Var(\hat{\beta})$
20	(2.0, 1.5)	(2.3172, 1.2835)	(0.7983, 0.8972)	1.4585	0.8074
	(0.5, 4.0)	(0.5132, 7.0972)	(0.9732, 0.9636)	0.0820	34.8920
	(4.0, 0.5)	(4.8230, 0.4973)	(0.9201, 0.9930)	7.8013	0.0723
	(2.0, 2.0)	(2.3672, 2.0874)	(0.8920, 0.9874)	1.3210	1.5672
30	(2.0, 1.5)	(2.3124, 1.2213)	(0.9230, 0.9972)	0.6707	0.4872
	(0.5, 4.0)	(0.5217, 6.0132)	(0.9731, 0.9835)	0.0631	23.7234
	(4.0, 0.5)	(4.6133, 0.4017)	(0.8937, 0.9083)	4.5983	0.0692
	(2.0, 2.0)	(1.9967, 2.0313)	(0.9538, 0.9876)	0.9012	1.5078
50	(2.0, 1.5)	(2.0891,1.1942)	(0.9074, 0.9920)	0.3948	0.2672
	(0.5, 4.0)	(0.4838, 5.0120)	(0.9235, 0.9574)	0.0572	13.0789
	(4.0, 0.5)	(4.6103, 0.3031)	(0.9318, 0.9927)	1.9071	0.0563
	(2.0, 2.0)	(1.9956, 1.9897)	(0.9536, 0.9897)	0.6752	0.9032
100	(2.0, 1.5)	(1.9762, 1.0701)	(0.9975, 0.9432)	0.3572	0.2014
	(0.5, 4.0)	(0.4702, 4.9673)	(0.9432, 0.9784)	0.0132	7.3210
	(4.0, 0.5)	(4.0259, 0.2123)	(0.9810, 0.9374)	1.8270	0.0513
	(2.0, 2.0)	(1.8130, 1.2704)	(0.9714, 0.9905)	0.3412	0.2715
150	(2.0, 1.5)	(1.8352, 1.0250)	(0.9512, 0.9930)	0.1327	0.0978
	(0.5, 4.0)	(0.3976, 3.9989)	(0.9618, 0.9568)	0.0115	0.2560
	(4.0, 0.5)	(3.9859, 0.2262)	(0.9805, 0.9907)	0.5098	0.0099
	(2.0, 2.0)	(1.7894, 1.1359)	(0.9758, 0.9853)	0.2081	0.2315

7. Concluding Remarks

In this paper, the various structural properties of the distribution are derived including explicit expressions for moments, mean deviation, Bonferroni and Lorenz curves, Renyi entropy and quantile function. The explicit expressions and recurrence relations for single and product moments of GOS are obtained from the GP distribution. The characterizing result of the GP distribution has been studied using conditional moments of generalized order statistics. The method of maximum likelihood is adopted for estimating the model parameters. For different parameter settings and sample sizes, the simulation studies are performed and compared to the performance of the GP distribution.

Acknowledgments

The authors are grateful for the comments and suggestions by the referees and the editor. Their comments and suggestions have greatly improved the article. Also first author grateful to Dr Neeraj Kumar, Amity University, Noida for his help and suggestions throughout the preparation of this paper.

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