STATISTICS IN TRANSITION new series, Spring 2014
Vol. 15, No. 2, pp. 243–268

MULTIOBJECTIVE OPTIMIZATION OF FINANCING HOUSEHOLD GOALS WITH MULTIPLE INVESTMENT PROGRAMS

Łukasz Feldman[1], **Radosław Pietrzyk**[2], **Paweł Rokita**[3]

ABSTRACT

This article proposes a technique of facilitating life-long financial planning for a household by finding the optimal match between systematic investment products and multiple financial goals of different realization terms and magnitudes. This is a multi-criteria optimization. One of the objectives is compliance between the expected term structure of cumulated net cash flow throughout the life cycle of the household with its life-length risk aversion and bequest motive. The second is financial liquidity in all periods under expected values of all stochastic factors. The third is minimization of net cash flow volatility. The fourth is minimization of costs of the investment plan combination. The result is a set of systematic-investment programs with accompanying information which programs are destined to cover which financial goal. Payoffs of one program may be used to cover more than one goal and the order may be other than sequential. An original goal function, constructed to suit conditions and assumptions of the proposed household financial plan model, is presented as an optimization procedure.

Key words: multiobjective optimization, personal finance, asset selection, intertemporal choice.

1. Introduction

Based on Modigliani and Brumberg (1954), Ando and Modigliani (1957) and Yaari (1965) life-cycle consumption, as well as on the dynamic asset allocation models by Merton (1969, 1971) and Richard (1975), a vast literature on lifetime financial planning for individuals has been developed so far. A common concept that underlies modern personal finance models is expressing intertemporal choice

[1] Wroclaw University of Economics, ul. Komandorska 118/120, 53-345 Wroclaw, Poland. E-mail: lukasz.feldman@ue.wroc.pl.
[2] Wroclaw University of Economics, ul. Komandorska 118/120, 53-345 Wroclaw, Poland. E-mail: radoslaw.pietrzyk@ue.wroc.pl.
[3] Wroclaw University of Economics, ul. Komandorska 118/120, 53-345 Wroclaw, Poland. E-mail: pawel.rokita@ue.wroc.pl.

situation in terms of expected discounted utility. Following Yaari (1965), the goal function to be maximized was expected discounted utility of consumption, where consumption was expressed as a consumption rate; utilities were weighted with conditional probability of survival of an individual, preferences did not change over time and were independent from period to period (time separable preferences). There was one argument of the utility function (consumption) and one utility function. This model was then developed and augmented in many directions. Bodie, Merton and Samuelson (1992) presented a model providing optimization of both consumption and investment decisions. Amongst other findings, they proposed to use consumption of leisure time (or, put it differently – the amount of work per unit of time) as an additional decision variable. They also showed the importance of human capital and its risk in consumption and investment decisions by individuals. On this ground a significant branch of personal finance models originated. Other assumptions of the original Yaari (1965) and Merton (1969, 1971) constructs were relaxed. The models allowed for habit formation (relaxing the assumption that preferences are independent in time), multiple risky assets (Bodie, at al., 2004), or optimization of retirement time (Sundaresan and Zapatero, 1997). Bodie (2007) presented a brief outline of the basic analytical framework including the most significant recent findings. Other propositions of further development include: using stochastic force of mortality in survival process (Huang, Milevsky, Salisbury, 2012), taking into account maximum psychological planning horizon (Carbone, Infante 2012) or behavioural biases – the concept of using hyperbolic discounting is included here (Ainslie, 1975, 1991; Kirby and Herrnstein, 1995). Geyer, Hanke and Weissensteiner (2009) presented a model allowing for stochastic labour income and investment opportunities. Scholz and Seshadri (2012) proposed to treat health as a type of asset and "production of health" as a particular form of investment. A lot of work has been also done in the area of retirement capital deployment in the retirement phase of the life cycle (Huang and Milevsky 2011; Milevsky and Huang, 2011; Gong and Webb, 2008; Dus, Maurer and Mitchell, 2004).

Despite many-sided and rapid development of the discipline, there are some important practical aspects of lifetime financial planning that have not been elaborated well yet. The aforementioned models concentrate on decisions made by individuals, whereas in personal finance a typical decision making entity is the household.

In this article a model of two-person household is used. A single decision maker is treated just as a special case.

The analysis of household consumption cannot neglect interconnections between persons. Even under the assumption that individual survival processes are independent, neither cash flows nor assets and liabilities assigned to household members are independent. For instance, cumulated investment in pension-plan products may be inherited by one spouse if that other dies before her (his) retirement age. The model assumes that after retirement date life annuity is bought, which, in turn, cannot be inherited. Thus, cumulated investment of one person depends on whether that other is alive, and if not – on time of death (in relation to retirement age). There are more such interdependences between financial categories building up consumption of the household. Moreover, some of the quantities are of cumulative nature. Also, the main household financial situation indicator used here, namely – the cumulated net cash flow (cumulated surplus), is a process of this kind. This makes conditional probabilities of being alive far insufficient for calculation of expected discounted utility. The whole history of the process needs to be taken into consideration instead. However, because the number of two-person survival process trajectories grows fast with the number of future periods spanned by the plan, some simplifications are needed. This is what was not necessary in the models discussed before. A proposal of simplification, which, moreover, has a very natural and practical interpretation, is presented here as one of the inputs.

Another specificity of the household, as opposed to single person, lies in the nature of risk connected with lifetime uncertainty. For a single person, only unexpected longevity might have adverse financial consequences. Thus, lifetime risk was regarded identical with longevity risk. For the household, also early death of one member (particularly the one who earns more) may threat financial liquidity.

One more difference is in retirement planning. It is not necessary (though most secure) that retirement income of each household member covers to the full extent fixed costs of the household and the part of variable costs that may be assigned to this person – 2 x full retirement as defined by Feldman, Pietrzyk and Rokita (2014b). The possibility of other retirement schemes (full-partial or even 2 x partial) gives a bigger range of feasible combinations of (1) the proportion of means allotted for consumption and investments, and (2) proportion of common investment assigned to particular persons.

Like in vast part of the literature, it is assumed here that financial goals set by the household are not subject to automated optimization. A satisfactory offset between the most desired and feasible bundle of goals (taking into account time structure of goals and their size) is approached recursively by means of external adjustments – if previous version of goal settings turns out to be unattainable given other constraints. The decision which goals should be rescheduled, reduced in size or abandoned is always made by household members. This approach is adopted because it would be a very hard task to define hierarchy of more than two goals, preferences of which may be not separable, nor transitive.

As far as the intended result of optimization is concerned, there is a difference between the majority of models discussed in the literature and the proposition presented here. The typical approach is focused on smoothing consumption, whereas in this research the aim is to obtain such term structure of cumulated surplus which best suits lifetime risk aversion and bequest motive of the household. Dependent on risk aversion level, financial plans differ just in shapes of expected cumulated surplus trajectory. The shape indicates which retirement profile will be realized (i.e., whether it will be 2 x full, full-partial or 2 x partial retirement). Concentrating on the shape of the cumulated net cash flow trajectory instead of consumption is a consequence of the way the household financial situation is modelled. The expected trajectory of the cumulated net cash flow is a fingerprint of each particular financial plan.

In addition to the aforementioned two-person household approach, intuitive and easily applicable definition of risk aversion measures and simplification of the optimization problem by limiting the number of survival scenarios, also the value function construction and its application may be ranked amongst original inputs of this research. The value function evaluates utility of the term structure of cumulated surplus, taking also into account consumption. It may be used as a goal function of the optimization procedure, but also – as it is discussed in more details in Section 4 – facilitates comparison of otherwise hardly comparable investment products. This property makes it a useful tool of finding a match between multiple investment products and household goals.

The paper is organized as follows. In Section 1 assumptions and basic components of the model are described. Household goals are discussed in Section 2. Construction of value function (being also a goal function for the optimization procedure) is presented in Section 3. It is a function of utilities calculated for consumption and bequest. What is also proposed is a simplification that allows avoiding taking all possible trajectories of bivariate survival process into account when searching for the maximum expected discounted utility. This simplification implies, at the same time, a straightforward definition of risk aversion measure (in respect of length-of-life related risk). Section 4 contains a step-by-step description of the procedure of matching multiple systematic-investment programs with a number of financial goals. Section 5 presents a numerical example. These are the results of the procedure described in Section 4 applied for a demonstration-case household. The last section concludes.

2. Basic concepts

When constructing a financial plan with a number of goals and multiple investment products available, two tasks are to be discussed. The first is selecting a combination from amongst available systematic-investment programs. The second is optimization of the term structure of household cash flows. While the question in the first task is which goals should be financed with which programs (assuming that one program may be used for financing more than one goal), in the second task the issue concerns the level of consumption, investments, as well as proportions in which household members participate in joint investments of the household, given all constraints, amongst which budget constraint is the most typical example. The procedure of carrying out the first task is described in Section 4. In the second task the goal function described in Section 3 is maximized. Both tasks are, of course, strictly connected. Combinations of investment programs chosen in preliminary selection as part of task 1 are evaluated by putting them into the model of household cash flows used in task 1 and calculating goal function value for each of them. The household cash flow model, together with the corresponding value function – being also the goal function of the optimization procedure in task 2 – is the basic construct to be discussed here. It is also the tool supporting financial plan construction (and, thus, among others, also carrying out task 1). In this section assumptions of the household cash flow model are provided and some basic notions shading light on how the model is constructed are discussed.

2.1. Assumptions

The model is based on a set of assumptions. They refer to the household itself, its incomes and expenses, construction of household goal function, and also to some elements of economic environment. They are as follows:

- Two-person household – if there are any persons other than the two *main household members*, they are represented in the model as elements of financial situation of the main two; a single person is treated just as a special case of a (reduced) pair.
- Both main members intend to remain in the household until their death.
- Goal function of the household is composed of two elements:
  - utility of consumption,
  - utility of bequest.
- Goal function s constructed using the concept of expected discounted utility.
- Survival processes of two main household members are independent.
- Joint utility function of the whole household is considered.
- Analytical form of utility function is the same for consumption and bequest.
- Household income in pre-retirement period is constant in real terms (inflation indexed).
- Fixed real rate of return on private pension plan.
- Pension income constant in real terms (inflation indexed).
- Fixed replacement rate (but may be different for women and men).
- Household members buy life annuity.
- Household consumption is fixed at the planned level unless running out of cumulated surplus (loosing liquidity).
- Optimization scope (not to be confused with domain of decision variables) is limited to the area determined by risk aversion of household members – *range of concern*.
- Risk aversion is limited to the length of life of household members; no other types of risk are considered.
- The surplus over consumption just cumulates – it is not invested, neither is it squandered.
- No will to work after retirement is taken into account. Thus, human capital in retirement is zero and the whole capital of the household that may be then

considered is reflected, on the asset side, just in form of cumulated investments into private pension plans and cumulated financial surplus.
- Bequest is not counted among financial goals of the household (but, if there is nonzero bequest motive, bequest-leaving potential is positively valued by the goal function).
- There is a constraint that all financial goals, together with retirement (that is, the main goal), must be realized – under expected values of death time no unutilized cumulated investment is left.
- Conditional survival probabilities used in discounted expected utility calculation may be obtained from any survival model, like Gompertz-Makeham (Gompertz 1825, Makeham 1860). This is, however, a secondary issue at this stage of research since the choice of mortality model does not influence in any way the very concept of the financial plan optimization procedure that is proposed here. It may, certainly, have impact on the final results of the optimization, which may require some detailed investigation at the later stage of the research (namely, when stability of the proposed model is be tested).

2.2. General characteristics of household cash flow model and consumption-bequest optimization

The largest and no doubt the most complicated building block of the whole financial plan optimization model is the model of household cash flows, also referred to as *household consumption model*. Its integral part is a value function reflecting preferences of the household. It is used as a goal function in household cash flow optimization. The value function is described in more details in Section 3, whereas main characteristics of the model are presented below and in Subsection 1.3.

- **Decision variables**

Decision variables with respect to which plan is optimized are: (a) proportion between consumption and investments and (b) division of total investments of the household between the two main household members. Proportions of the two persons in total investments are important. This is, among others, because life annuity of one person vanishes with death of this person and cannot be inherited. If the person who had bought higher life annuity died first, it would have much more severe consequences for the household finance than if it was the person who had bought lower annuity.

- **Incomes, consumption, investments**

The model is based on consumption utility, but the main diagnostics of financial situation throughout the whole life cycle of the household is the cumulated net cash flow (cumulated surplus). This is because the financial plan assumes preservation of some predefined standards of living. This means constant consumption in real terms (or constant growth of consumption in real terms).

Thus, consumption is not necessarily the whole difference between incomes and investments. The main types of cash flows in the model are: (a) basic incomes (without investment liquidation, etc.), (b) costs (basic consumption), then – dependent on goals to be financed – also (c) cash flows resulting from pre-financing and post-financing of goals (investments, credit repayments, etc.). The difference between incomes and the sum of consumption and investments (and instalments) gives (d) the net cash flow. It cumulates over time. One of constraints imposed on financial plan is to secure household liquidity, thus not to let the cumulated net cash flow to fall below zero (the net cash flow of a given period may occasionally be negative if there is a potential to cover the shortfall in the future). Liquidation of investments, as well as transfer of credit capital to the household are additional incomes. Expenditures on realization of goals are additional elements of consumption, but they are treated separately from basic consumption. The separate treatment consists in calculating utility only of basic consumption. With a bequest motive, also the potential to leave bequest is taken into account in utility calculation.

- **Consumption-investment trade-off and risk aversion**

The decision about consumption determines the standard of live. The higher standard of life, other things unchanged, the lower value of the capital left unutilized. A need arises to find a trade-off between consumption in pre-retirement period and safety of consumption in retirement period. It is assumed that the sum of cumulated investments and cumulated net cash flow available after retirement must be sufficient to generate pay-offs that fill retirement gap. Retirement gap is understood as the difference between the last job income and retirement from compulsory public pension system. But the question for how long it should be sufficient is open to the decision of the household members. It depends on risk aversion of the household. A simple way of grasping the notion of longevity risk aversion is asking these persons how many years after the expected time of death of the one who is expected to live longer a potential threat of permanent financial shortfall seems too abstract to be a cause for concern.

- **Role of bequest motive**

The higher cumulated surplus the better protection against longevity (and also premature death) risk. On the other hand, leaving any surplus or unutilized investment after the last household member dies may make sense only if the household wants to leave a legacy to someone. Otherwise, it would be a suboptimal solution. The task of finding a trade-off between safety and economic efficiency of capital utilization will be different for the case with and without bequest motive. This difference would be particularly clear if the household showed no risk aversion at all. Then, for the case without the bequest motive the optimal plan would be such that its expected trajectory of cumulated surplus shrinks to zero at the date of the expected death of the last household member.

The household with no risk aversion, yet intending to leave some bequest, might, in turn, accept some unutilized capital at the end of their lifetime.

- **Changing parameters and constraints**

There are some quantities whose values may be changed in plan revision mode, which are not, however, decision variables. These are parameters and constraints that are subject to verification and adjustments by decision of the household. Main constraints that are adjusted in this way include financial goals. On the one hand goals must be met fully and on time. On the other hand, if this condition comes out to be infeasible (negative values in any point of the expected cumulated surplus trajectory), then goals are revised.

2.3. Input and output

Listed below are input and output arguments of the household consumption model. An important part of the model is the cash flow optimization procedure based on goal function described in Section 3. For cash flow optimization, the starting values of decision variables are the main input. The output comprises: the optimum values of decision variables, the maximum of the goal function obtained as a result, and the expected trajectory of cumulated net cash flow for the optimal solution. The decision variables are: assumed consumption at the moment t_0 (C_{a_0}) and proportion of investments in private pension plans by Person 1 and Person 2 ($\upsilon_1, \upsilon_2 = 1 - \upsilon_1$). Apart from decision variables all initial values of variables and all parameters of the household cash flow model are certainly also the input of the optimization procedure.

- **Input:**
  - Age at t_0: $x_0^{(1)}, x_0^{(2)}$,
  - Retirement age: $zR1 = z(R1;\ x_0^{(1)}), zR2 = z(R2;\ x_0^{(2)}$,
    where $R1$ and $R2$ are retirement dates, $z(t, x)$ is age at the moment t of a person who was x years old at t_0,
  - Expected length of life at t_0: $E\ \left(D\middle|z\left(t_0;\ x_0^{(1)}\right)\right), E\ \left(D\middle|z\left(t_0;\ x_0^{(2)}\right)\right)$,
  - Income at t_0: $Ic_0^{(1)}, Ic_0^{(2)}, Ic_0^{(c)}$,
  - Income growth rate: g_1, g_2, g_c,
  - Replacement rate: χ,
  - Constant common consumption at t_0: FC,
  - Individual consumption at t_0: $VC_0^{(1)}, VC_0^{(2)}$,
  - Minimum acceptable consumption in any period: C_{min},
  - Consumption growth rate: $h_{FC}, h_1\ , h_2$),
  - Proportion of investments in private pension plans by Person 1 and Person 2: $\upsilon_1, \upsilon_2 = 1 - \upsilon_1$,
  - Return on investment: r_{Iv},

– Return on "uninvested" surplus: r_{Spl},
– If with other goals (other than retirement):
  • Other goals ($\boldsymbol{G}$),

  where:

$$\boldsymbol{G} = [G_1, \dots, G_n] = \begin{bmatrix} T_1 & \dots & T_n \\ M_1 & \dots & M_n \end{bmatrix},$$

  $G_j = \begin{bmatrix} T_j \\ M_j \end{bmatrix}$ – j-th goal (denoted also as: $G_j = (T_j, M_j)$),
  T_j, M_j - planned time and magnitude of goal j,

  • Available investment programs for financing other goals than retirement ($L = [L_1 \quad \dots \quad L_m]$).
  • Information about assignment of goals to financial programs (for explanation why it is input and not the output of the cash flow optimization task, refer to Section 4);

- **Output:**
  1) Direct:
  – Trajectories of consumption process,
  – Trajectories of surplus process,
  2) Indirect:
  – Income process,
  – Consumption process,
  – Cumulated investment process;

- **Relationships between some chosen input positions and basic household cash flows:**

  Consumption may be divided into three components:
  – Common consumption (fixed and not attributed to any particular person),
  – Consumption of Person 1,
  – Consumption of Person 2.

  Division of consumption between household members is vital for determining their contributions to private pension investment programs. The programs are separated and they do not depend on each other, however, if a person dies before retirement age, the amassed capital is transferred to the other one.

  Total consumption and savings of the household are given as (eq. 1, 2):

  ▪ Assumed consumption:

$$C_{a_t} \equiv VC_t^{(1)} + VC_t^{(2)} + FC \tag{1}$$

  where: C_{a_t} – assumed consumption, $VC_t^{(i)}$ – variable costs assigned to i-th person, FC – fixed costs of the household.

- Savings (difference between incomes and consumption):

$$S_t = Ic_t - C_{a_t} = Ic_t^{(1)} + Ic_t^{(2)} + Ic_t^{(c)} - VC_t^{(1)} - VC_t^{(2)} - FC \qquad (2)$$

where: Ic_t – joint income at the moment t, $Ic_t^{(1)}$ – income of the first person, $Ic_t^{(2)}$ – income of the second person, $Ic_t^{(c)}$ – income of the household that is not assigned to any person (e.g.: an income from renting out a real estate being a part of conjugal community).

Under the assumptions of the model, consumption needs are fixed or deterministically dependent on the life-cycle phase. Income, whether from labour or retirement, is either consumed or, in part that exceeds consumption needs, constitutes unconsumed and uninvested surplus. It is certainly also possible that the income of a given period does not cover consumption needs.

- Surplus – uninvested part of savings of a given period (eq. 3):

$$\begin{aligned} NCF_t &= S_t - Iv_t = Ic_t - C_{a_t} - Iv_t = \\ &= Ic_t^{(1)} + Ic_t^{(2)} + Ic_t^{(c)} - VC_t^{(1)} - VC_t^{(2)} - FC - Iv_t^{(1)} - Iv_t^{(2)} \\ &\quad - Iv_t^{(c)} \end{aligned} \qquad (3)$$

$$\begin{aligned} &(Ic_t = Ic_t^{(1)} + Ic_t^{(2)} + Ic_t^{(c)}; \\ &Iv_t = Iv_t^{(1)} + Iv_t^{(2)} + Iv_t^{(c)}; \\ &\text{if } t > R_i \text{, then } Ic_t^{(i)} = Icb_t^{(i)} + Icc_t^{(i)}) \end{aligned}$$

where: Iv_t – investments of the household in period t, $Iv_t^{(1)}$ – investments of the first person in period t, $Iv_t^{(2)}$ – investments of the second person in period t, $Iv_t^{(c)}$ – investments of the household that are not assigned to any person in period t; moreover: $Icb_t^{(i)}$ – i-th person retirement income from a public pension system (all pillars included), $Icc_t^{(i)}$ – i-th person retirement income from private pension plan(s), R_i – retirement date of person i.

- Cumulated surplus – cumulated net cash flow (eq. 4):

$$CNCF_t = \sum_{\tau=0}^{t-1} NCF_\tau \qquad (4)$$

- Maximum feasible consumption (eq. 5):

$$C_{ft}^* = Ic_t - Iv_t \qquad (5)$$

(no cumulated surplus would be generated then because surplus of a given period would be consumed).

- Consumption that can be actually realized at a given moment t, assuming that until the moment only the assumed consumption was realized (eq. 6):

$$C^*_{f_t} = Ic_t + CNCF_t - Iv_t \tag{6}$$

- Consumption as understood in this model (i.e. assumed consumption, but up to the level that may be actually realized (eq. 7):

$$C_t = \min\{C_{a_t}, Ic_t + CNCF_t - Iv_t\} = \min\{C_{a_t}, C^*_{f_t}\} \tag{7}$$

or equivalently (eq. 8):

$$C_t = C_{a_t} + \min\{0, CNCF_t\} \tag{8}$$

In the formulas 6 and 7 there are no direct references to any further detailed decomposition of costs, incomes and investments. But it is important, after all, to be able to recognize individual contribution of each person to the total net cash flow of the household. This allows modelling the impact of stochastic elements of the model (namely, of the dates of person 1 or 2 deaths – $D1$ and $D2$).

3. Financial goals

Besides consumption sustaining, ensuring realization of the goals is the reason for which the financial plan is constructed. The goals differ in size, timing and other characteristic. In this section retirement-type goals and other goals are discussed.

3.1. Main financial goals of the household

The basic version of the model assumes only two main financial goals: retirement and bequest. These two goals have their unique feature – they cannot be post-financed. Therefore, the only way to realize them is to build up sufficient capital over the years. Retirement capital, as well as bequest capital, are usually very high in comparison to monthly income of the household. Thus, the earlier saving and investing are started the better.

The classical approach to consumption optimization assumes that: a) retirement income of the household should be at least as high as total consumption of the household, and b) individual retirement income of the household member should not be lower than his or her individual financial needs in retirement. This approach would be safe indeed, but rather inefficient due to overlapping coverage of household fixed costs, resulting in a considerable unutilized surplus. Neglecting this surplus would, in turn, lead to overestimating retirement capital needs and, consistently, paying unnecessarily high contributions to private pension plans in pre-retirement period. It is possible to propose such investment mix that would be less expensive than traditional approach, but would result in more risky retirement income. After all, it should be chosen so that it is suited to preference structure of household members. Generally speaking, the solutions differ in how much of household fixed and variable costs is covered by retirement income and in what proportions household members participate in them (comp. Feldman, Pietrzyk, Rokita, 2014a).

Emphasis should be also put on building the capital for bequest. It is modelled here as cumulated surplus (comp. eq. 4) that remains at the time of death of the last household member.

Taking bequest motive into consideration is necessary in this approach since the consumption may be the same in plans that differ much in respect of the term structure of cumulated surplus, and thus – financial situations of the household. Only shortfalls in cumulated surplus, driving consumption below its assumed value, would be visible for the utility of consumption. If, however, the last living household member dies, the uninvested and unconsumed surplus becomes visible in the form of bequest. Because this may happen with some probability at any moment, the value function takes account of cumulated surplus along the whole planning period.

These two above-mentioned financial goals are put in the centre of the model not only for their magnitude, but also because they usually are the last and often underestimated financial goals of the household. It is also obvious that the ability of achieving these goals depends significantly on household ability to save money. In most cases households spend their savings on some durable goods or other unplanned expenses, exchanging long term utility of sustaining high consumption level for short term utility of unplanned additional consumption (including realization of additional goals). However, households do not make a fully conscious choice in this respect. First of all, the decision makers often lack skills to estimate their retirement capital needs. Secondly, they often neglect the bequest motive and treat bequest as their estate. Thirdly, they are unaware of how the additional (unnecessary) expenditures affect their future financial situation.

3.2. Subordinate financial goals

There are two most commonly used approaches to determining financial goals: age based and life-event based (Nissenbaum, Raasch and Ratner, 2004). The first approach assumes financial goals are strictly dependent on the age of a decision maker. For instance, a 25-year old single male has different needs in terms of retirement planning than 40-year old male. The second approach focuses rather on the needs arising from particular events, determining some important elements of a decision maker's life situation. A single person usually has no bequest motive, while parents of two teenage children have a vital need to leave some legacy. Both approaches are justified to some extent and, in fact, result in similar outcome.

Apart from retirement and bequest, households have a wide variety of other financial goals. The most common include:

- Getting married,

- Buying a house,
- Raising a family,
- Funding education for children,
- Purchasing durable goods of high value.

Unlike retirement or bequest, all these financial goals may be post-financed. For the purposes of personal financial planning each financial goal has to be described by at least two characteristics: time of occurrence and value (assuming that goals are deterministic themselves). However, most decision makers do not have sufficient knowledge to determine these parameters. Firstly, due to lack of long-term planning (majority of individuals just do not plan). Secondly, because of insufficient information. Moreover, some of the parameters that need to be taken into consideration change in longer run. And they are in fact stochastic. For example, the question may arise of how real estate prices will change in the next 5 years.

It is important to point out a very significant difference between expenditures on the goal "child" (or "children") and funding education for children, on the one side, and realization of other financial goals, on the other. In most cases realization of a financial goal is an event occurring at particular point in time (e.g., purchase of a house, car, etc.). When it comes to raising children, the "realization" of that financial goal lasts in time. Therefore, it is assumed here that expenditures associated with children are treated as an increase in consumption. In order to stay in conformity with equation 8, these expenses increase the basic consumption by particular percentage (given as a model parameter) as long as the child remains in the household.

4. Goal function of the household

The goal function (value function) is intended to take into account both utility of consumption and utility of bequest (unconsumed and uninvested cumulated financial surplus). Utility function used for consumption and bequest is identical; just different arguments are put in. Apart from probabilities and discounting factors, these component utilities are multiplied by factors depending, among others, on attitude towards risk and bequest motive. The key concept of the goal function definition lies in these multipliers. One period value function is the weighted sum of component utilities. The goal function for the planning period has a form of expected discounted utility.

4.1. Utility

Household utility is split between the utility of consumption $U(C)$ and utility of bequest $U(B)$.

Moreover, utility of consumption is divided in two parts, with respect to time: the period before and after the expected death (see point 3.2).

Since the surplus, by definition, is what has not been consumed, it is not taken into account by utility of consumption. The surplus cumulated in previous periods may be partially consumed if current incomes are not sufficient to cover current expenses, but the utility of consumption does not recognize the sources from which the consumption is financed. This is why utility of consumption alone would be insufficient in this model.

Given other conditions and constraints (like financial goals and their financing) unchanged, the higher consumption the lower surplus left to build up the cumulated net cash flow. Since these two aspects of the financial plan are strictly contradictive, there has to be some trade-off between them. The trade-off is expressed with the following weights (eq. 9):

$$\alpha = 1 - \beta \tag{9}$$

where:

α – consumption preference parameter, β – bequest preference parameter.

Furthermore, the intertemporal consumption choice demands to discount the utility at some rate r_C. It is obvious that the utility of bequest should be also discounted, but at some other rate r_B. The relation between these rates should be (eq. 10):

$$r_B < r_C \tag{10}$$

The discount rate for the bequest has to be smaller because the household can postpone the realization of the bequest motive or even give it up, while the consumption at minimal level has to be achieved.

4.2. Risk aversion and optimization area

As it has been mentioned in the introduction, not all scenarios of the survival process are taken into account. The modification of the way the survival of two persons is worked into the model is twofold. The main concept of the simplification consists in considering only some periods before and after the expected time of death. This delineates a range in which premature death or unexpected longevity is recognized to be a concern for the household members. The *range of concern* defined in this way will be set in accordance to life-length risk aversion. In this approach optimization for a single person would be performed for the values of potential death time from within the interval of (eq. 11):

$$D_i^* \in [E(Di) - \gamma^*; E(Di) + \delta^*] \tag{11}$$

It is worth emphasizing that the range of concern should not be confused with domain of optimization, because it is not a set of decision variable values.

The second simplification is in probabilities used. Survival probabilities are not conditional probabilities for a given day but unconditional probabilities (conditional under the condition of surviving until the moment t_0). The second simplification is – from the point of view of the main idea – just a side issue, and may be refrained from in further stages of the research. One just needs to remember that when attempting to make the model more dynamic the whole history of the survival process would need to be considered for each scenario and each period (not just the state of the household in the preceding period). This is due to complicated interdependences between quantities used for cash flow calculation and cumulative nature of the net cash flow process.

For the household, the range of concern is a rectangle of (eq. 12):

$$Range_{Hh} = [E(D1) - \gamma^*; E(D1) + \delta^*] \times [E(D2) - \gamma^*; E(D2) + \delta^*] \qquad (12)$$

where:

γ^* – premature death risk aversion parameter (number of years that the household takes into consideration),

δ^* – longevity risk aversion parameter (also interpreted as the number of years),

There is one parameter γ^* and one δ^*, characteristic of the household, not individual person.

On the basis of risk aversion parameters (γ^* and δ^*), risk aversions measures, $\delta(t)$ and $\gamma(t)$, are constructed. These are then used as multipliers by which utility of consumption for periods before and after the expected time of the end of the household is multiplied. They are defined so that the premature-death risk aversion multiplier is the higher the earlier moment before the expected time of the end of the household decreases to 1 at the expected time of the end of the household, to decay afterwards, whereas the longevity risk multiplier reaches unity at the expected time of death and then increases with time. A proposed formal definition that holds these properties is given by eq. 13 and 14:

$$\gamma(t) = \begin{cases} \left(\frac{1}{1+\gamma^*}\right)^{\left(\frac{t-E(D)}{E(D)}\right)} & t \leq E(D) \\ 0 & t > E(D) \end{cases} \qquad (13)$$

$$\delta(t) = \begin{cases} (1+\delta^*)^{\left(\frac{t-E(D)}{\delta^*}\right)} & t > E(D) \\ 0 & t \leq E(D) \end{cases} \qquad (14)$$

where:

$E(D)$ is unconditional expected time of the end of the household, defined by eq. 15:

$$E(D) = max(E(D1), E(D2)) \qquad (15)$$

and $E(Di) = E(Di|Di > t_0)$ is unconditional expected time of death of Person i.

One of the merits of defining lifetime risk aversion in the form of γ^* and δ^* is that these parameters do not require estimation nor detailed inquiry. Their interpretation seems to be sufficiently intuitive for household members just to declare their values.

The optimization procedure differs significantly from the most commonly used ones. In classical approaches consumption is optimized across the whole life cycle of a decision maker. But that might result in excess saving and amassing too much retirement capital. The household would have to decrease its consumption in early years in order to fulfil optimization constraints at every point in time and for each combination of individual survival scenarios, even for those very unlikely (e.g., a man dies at the age of 25 and his wife lives up to 95 – possible, but it would be very likely that the young widow would find another lifetime partner and raise a new household for which the old financial plan would be utterly irrelevant). The model presented here focuses on the range of concern that corresponds to probabilities recognized arbitrarily by both decision makers as significant. Secondly, optimization over the range of all possible combination of dates when household members may die is very computationally-intensive. The number of possibilities increases proportionally to the square of the number of years taken into account.

4.3. Goal function

The goal function used here is based on the concept of expected discounted utility of consumption. It differs, however, from the one used in classical life cycle models, like that of Yaari (1965). It has two components: the first one is responsible for utility of consumption and the other reflects utility of unconsumed surplus (bequest). Both are joint utilities of the whole household. This is a necessary condition if one common life-long financial plan is to be constructed.

The goal function presented in eq. 16 is an expansion of that proposed by Feldman, Pietrzyk and Rokita (2014a). It is suited to the model of two-person household with rectangular range of concern.

$$V = \sum_{D_2^*=E(D2)-\gamma^*}^{E(D2)+\delta^*} \sum_{D_1^*=E(D1)-\gamma^*}^{E(D1)+\delta^*} p \left[\begin{array}{c} \alpha \left(\sum_{t=0}^{max\{D_1^*,D_2^*\}} \frac{1}{(1+r_c)^t} u\big(C(t;D_1^*,D_2^*)\big)\big(\gamma(t)+\delta(t)\big) \right) + \\ \beta \frac{1}{(1+r_B)^{max\{D_1^*;D_1^*\}}} u\big(B(max\{D_1^*,D_2^*\};D_1^*,D_2^*)\big) \end{array} \right] \to max \qquad (16)$$

where:

$u(.)$ – utility function (the same in all segments of the formula),

$C(t;D_1^*,D_2^*)$ – consumption at the moment *t*,

$B(t;D_1^*,D_2^*)$ – bequest (cumulated investments and surplus of both household members at the moment *t*,

$\gamma(t)$ – premature death risk aversion measure (depends on γ^*),

$\delta(t)$ – longevity risk aversion measure (depends on δ^*),

p – probability that at least one person is alive,

r_C – discount rate of consumption,
r_B – discount rate of bequest.

5. Technique of financing household's goals

The main purpose for which households prepare their financial plans is to achieve all their financial goals. Due to a wide variety of ways in which the goals may be financed and the fact that their examination and comparison is a complex task, households prefer ready-made investment products. One of the most common forms is systematic investment plans offered by mutual funds.

This solution is noticeably elastic and may be used for financing every financial goal. Although systematic investments require some discipline and may seem psychologically hard, they have many advantages over post-financing. Only part of the financial goals may be financed by debt (housing, cars, etc.). Moreover, taking loans is often more expensive (though easier in many respects). Households also may face limitations according to their credit standing.

The assumption that all goals must be realized is still sustained. Households usually take into consideration only the goals that are planned for the near future. Such goals as retirement or bequest are neglected (more or less on purpose). Such attitude affects significantly household's ability to realize all its financial goals. It may even make some of them unattainable. It is recommended that household members work all important goals they have into their financial plan.

That being so, the household faces the problem of how to finance *n* different goals when *m* possible investment programs are available. The problem is multidimensional not only because of the number of financial goals, but also for wide variety of parameters to be taken into consideration (i.e., rates of return, indexation, fees and charges, allocation rate, etc.).

It is proposed here to facilitate this process with an algorithm that seems to be indeed easy to understand and use. Such technique (algorithm) has to give a result that fulfils the following criteria:

- Expected term structure of cumulated net cash flow (obtained after application of the financing strategy selected by the algorithm) should be in compliance with life-length risk aversion and bequest motive.
- Financial liquidity of the household must be sustained.
- Net cash flow volatility is minimized.
- Costs of the investment plan combination are minimized.

We also assume that one program may be used to provide cash to cover more than one goal and the order may be other than sequential.

For illustrative reasons, let us assume that the household has three financial goals and there are three different investment programs available on the market. Each of the programs reflects the same level of risk. The household wants to find an optimal set of investment programs along with information which program is destined to cover which financial goal.

The investment mix selection is a multi-step process, whose three main steps are:

1) Minimization of contribution to investment programs.
2) Selection of effective investment mix.
3) Cash flow term structure fitting.

Before an analysis of concrete systematic-investment products is started, *general schemes* of financing are identified. A general scheme may be defined as a $(2 \times n)$ matrix $\boldsymbol{S}$, in the first row which includes indices denoting goals, whereas the second row is constructed using the following algorithm:

a) put any symbol, say "A", in the first field of the second row of matrix $\boldsymbol{S}$: $S_{2,1} = A$,
   a general way of financing goal 1 is then: $S_{:,1} = \begin{bmatrix} 1 \\ A \end{bmatrix}$;

b) for $j := 2\ to\ n$ repeat the following:
   c.1) if goal j is to be financed with the same program as some of the goals considered so far (i.e. goals: $[1, \ldots, j-1]$), let it be a goal υ, $(1 \le \upsilon \le j-1)$, whose general way of financing is: $S_\upsilon = \begin{bmatrix} \upsilon \\ \Upsilon \end{bmatrix}$,
   then substitute
   $S_{2,j} = \Upsilon$,
   a general way of financing goals $[1, \ldots, j]$ is then: $S_{:,[1:j]} = \begin{bmatrix} 1 \cdots \upsilon \cdots j \\ A \cdots \Upsilon \cdots \Upsilon \end{bmatrix}$.
   c.2) otherwise (goal j is intended to be financed with another program), assign a symbol that has not been used yet, say Ξ;, to goal j; thus, substitute:
   $S_{2,j} = \Xi$,
   a general way of financing goals $[1, \ldots, j]$, is then: $S_{:,[1:j]} = \begin{bmatrix} 1 \cdots \upsilon \cdots j \\ A \cdots \Upsilon \cdots \Xi \end{bmatrix}$;

For 3 goals and 3 programs 4 general schemes of financing are possible:

$$\boldsymbol{S1} = \begin{bmatrix} 1\,2\,3 \\ AAA \end{bmatrix},\ \boldsymbol{S2} = \begin{bmatrix} 1\,2\,3 \\ AAB \end{bmatrix},\ \boldsymbol{S3} = \begin{bmatrix} 1\,2\,3 \\ ABB \end{bmatrix},\ \boldsymbol{S4} = \begin{bmatrix} 1\,2\,3 \\ ABC \end{bmatrix}.$$

It should be stressed that sequences ABB and ACC are identical. The same refers, for example, to sequences AAB and AAC. General schemes inform whether goals are to be financed by the same or different programs, not determining, yet, which concrete programs might there be.

Then, the investment mix selection algorithm is executed.

In the first step (*Minimization of contribution*) we find the minimum contribution for each possible combination of goals and available investment programs. Due to incomparability of cash flow term structures of different investment programs, contribution is minimized for each possible general program scheme separately. Investment mix for a given general scheme that minimizes contribution is further on referred to as *efficient investment mix*.

Table 1. Types of investment schemes. Scheme depends on the number of different investment programs used to finance goals and the structure of which program is used to finance which goal

INVESTMENT MIX	GOAL 1	GOAL 2	GOAL 3	SCHEME
Investment mix 1	**Program 1**	**Program 1**	**Program 1**	AAA
Investment mix 2	Program 2	Program 2	Program 2	
Investment mix 3	Program 3	Program 3	Program 3	
Investment mix 4	Program 1	Program 1	Program 2	AAB
Investment mix 5	Program 1	Program 1	Program 3	
Investment mix 6	**Program 2**	**Program 2**	**Program 1**	
Investment mix 7	Program 1	Program 2	Program 2	ABB
Investment mix 8	**Program 1**	**Program 3**	**Program 3**	
Investment mix 9	Program 2	Program 1	Program 1	
Investment mix 10	**Program 1**	**Program 2**	**Program 3**	ABC
Investment mix 11	Program 2	Program 3	Program 1	
Investment mix 12	Program 3	Program 1	Program 2	

In the second step (*Selection of efficient investment mix*) efficient solutions are picked. A solution is said to be efficient if it requires the minimum contribution to finance household goals from amongst investment program combinations belonging to the same general scheme (compare Table 1).

In the third step (*Cash flow term structure fitting*) efficient solutions selected at the second stage are put into to the model of household cash flow. The optimal solution is such that the corresponding term structure of the cumulated net cash flow of the household best fits the preferred one. Two alternative approaches to evaluate the fit are proposed:

A. least squares method,
B. maximization of goal function by putting each of the efficient investment mixes into the model of household cash flow discussed in Sections 1-3, finding the value of the goal function (compare Section 3) for each of them, and picking up the one that maximizes the goal function maximum.

The optimal solution has to meet the following conditions: (1) consumption has to be higher than the minimum (set by the household), (2) at any point in time there has to be some nonnegative cumulated surplus, (3) goal function is to be maximized.

6. Numerical example

The following example shows in work the results of the algorithm described above. Let us assume that the future cumulated surplus term structure of the household (for the next 30 years) is given as below (Figure 1 a):

Cumulated surplus term structure before the financial goals

Figure 1. a)

Cash flow term structure of financial goals

Figure 1. b)

Cumulated surplus term structure after realization of financial goals

Figure 1. c)

Figure 1. Cumulated surplus term structure of the household before and after realization of financial goals

The household has two financial goals that are planned to be achieved in year 9 and 18. The size of these goals is known (or can be easily estimated) (compare Figure 1 b).

If the household decides to realize its financial goals from cumulated surplus, then the final term structure of cash flow will look like in the Figure 1 c.

The household may, however, invest some part of its surplus into an investment program. Let us assume that there are two programs available on the market. That gives two possible schemes to be analyzed (Figure 2 a, Figure 2 b). The first (scheme AB) uses two programs separately to finance two goals, and the second (scheme AA) uses one program to finance both goals. Negative cash flows in Figure 2 a and Figure 2 b are contributions to investment programs. Positive ones are pay-offs from the programs.

Realization of two financial goals with two different investment programs (scheme AB)

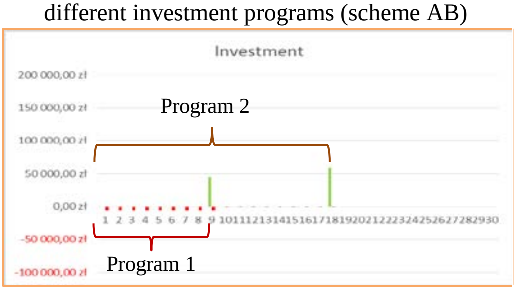

Figure 2. a)

Realization of two financial goals with one investment programs (scheme AA)

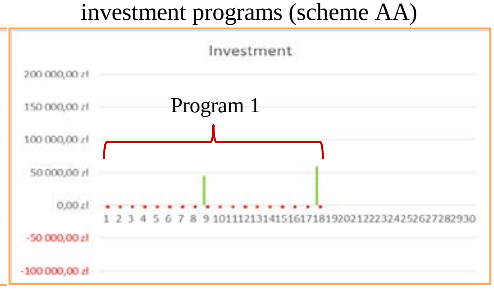

Figure 2. b)

Cumulated surplus term structure comparison

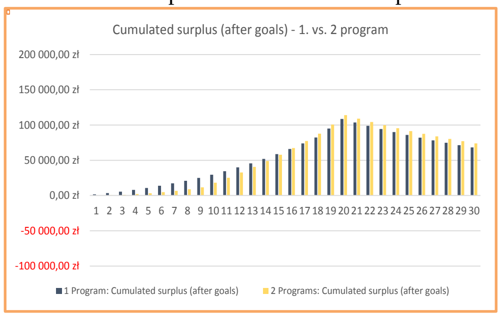

Figure 2. c)

Figure 2. Impact of different investment schemes on cumulated surplus term structure

Dependent on the scheme one uses, different cash flow term structures are obtained. The comparison of these structures is presented in Figure 2 c)

Then, both structures are compared with the optimal trajectory and the final result is given.

The optimal trajectory might be estimated for strategic asset allocation that reflects the risk level of investment programs, but not concrete programs themselves. Then the term structure of cash flow is calculated and compared to the cash flow term structure resulting from investment in particular efficient investment plan.

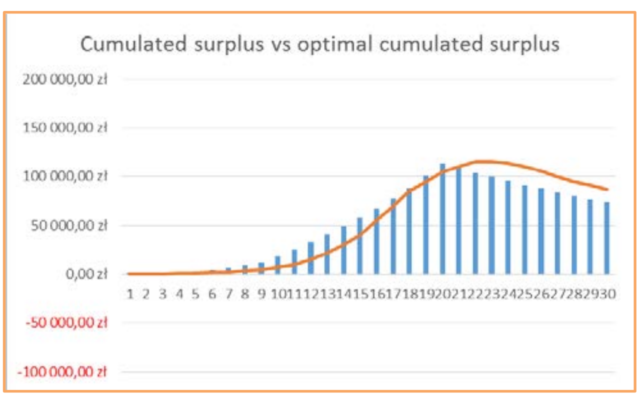

Figure 3. Cumulated surplus comparison with optimal trajectory

Another approach would be just calculating the value of household goal function for both program mixes and selecting the one with higher value. All conditions have been listed at the end of Section 4.

7. Technical issues

The formula of goal function (eq. 16) presented in section 3 does not specify in details the analytical form of the utility function. In the numerical example discussed in this article a square root utility was used. This was, however, modified in such a way that it took on value zero for scenarios in which cumulated surplus fell below zero at any point in time. This, certainly, does not need to drive the goal function to zero because the goal is a sum of probability-weighted discounted expected values of utility for all scenarios within the range of concern. The argument for such solution is that a scenario cannot be "partially" satisfactory if it guarantees high level of consumption in some period and then leads to permanent shortfall (i.e. practical bankruptcy of the household). Within the bunch of scenarios there may by one or more such zero-utility scenarios. Their influence on the goal function depends on their probabilities.

Such construction of the goal function causes some technical inconvenience. The goal function becomes indifferentiable on vast parts of its domain. Moreover, there are not only unsmooth jumps in its value, but also local extremes. There is, however, a simple way to overcome this problem without reaching for very advanced optimization techniques. The goal function shows problematic properties mostly along one dimension, namely – the decision variable describing division of total investment between household members. Along the second one, that is consumption-investment proportion, it behaves in a much more conventional way. It is continuous, differentiable and unimodal up to the maximum, though indiferrentiable and showing local extremes after reaching the global maximum.

Along the first dimension (division of investment contributions) the function is sliced into a finite number of cross-sections. The range between 100% and 0% of the total household investment allocated to Person 1 may be divided into any number of scenarios. Then, for each of the slices a maximum along the second dimension is searched for.

It may be observed that the cross-section of the goal function along the second dimension (consumption-investment) always shows the following property: it is differentiable and increasing until it reaches global maximum for the given slice, then a downwards jump is encountered and then there may be a local maximum (always lower than the first maximum - walking from the left - for this particular slice), followed with a rapid drop. At this stage of analysis, continuous optimization may be used under the condition that the optimization algorithm starts from the lowest values of consumption and searches for the maximum of the goal in the direction of growing consumption.

Then, the maximum of maxima for each slice is taken as the global maximum for the whole goal function.

8. Summary

The model presented here involves some original approaches and solutions, and sheds some new light on household consumption optimization. It focuses on the household, not on a single decision maker only. The optimization area is strictly dependent on the risk aversion of household members and is narrowed to the most probable scenarios. This results in higher optimal consumption for the household than it would be derived from models taking the whole lifespan of decision makers into account. The risk aversion measures are very intuitive and their interpretation, calibration and use by decision makers is straightforward. Different discounting rates for utility are used. The same goal function as used in optimization model allows comparing different cash flow term structures. Thus, it may be used to facilitate choosing from amongst incomparable investment products.

Further research will focus on expanding its application to stochastic behaviour of financial goals. In particular, such goals as children should be treated in this way because of stochastic nature of a child's birth (conditional on planned time).

Furthermore, other types of risk than just risk related to length of life will be analyzed. In particular, risk connected with investments, mainly market risk (e.g.: interest rate risk, stock price risk, etc.) will be taken into consideration.

Together with taking account of investment risk, also risk of human capital will be a natural object of investigation. Adopting the approach by Bodie, Merton and Samuelson (1992), in which risk of human capital, increasing with age, is offset by decreasing riskiness of investments, may be useful in the next stages of research.

Also stability of the model will be analyzed. This is not only sensitivity of optimization results to changes of parameters that needs to be analyzed. What is also worth investigating is how the choice of the underlying survival process model will influence the final results.

Another area of research may be examining structure of hierarchy of financial goals and suggesting optimization procedures.

Acknowledgements

The research project was financed by The National Science Centre (NCN) grant, on the basis of the decision no. DEC-2012/05/B/HS4/04081.

REFERENCES

AINSLIE, G., (1991). Derivation of „Rational" Economic Behavior from Hyperbolic Discount Curves. The American Economic Review, 81(2), pp. 334–340.

AINSLIE, G., (1975). Specious reward: A behavioral theory of impulsiveness and impulse control. Psychological Bulletin 82(4), pp. 463–496.

ANDO, A., MODIGLIANI, F., (1957). Tests of the Life Cycle Hypothesis of Saving: Comments and Suggestions. Oxford Institute of Statistics Bulletin, Vol. XIX (May), pp. 99–124.

BODIE, Z., DETEMPLE, J., OTRUBA, S., WALTER, S., (2004). Optimal Consumption-Portfolio Choices and Retirement Planning. Journal of Economic Dynamics and Control, 28, pp. 1115–1148.

BODIE, Z., MERTON, R. C., SAMUELSON, W. F., (1992). Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model. Journal of Economic Dynamics and Control, 16(3-4), pp. 427–249.

BODIE, Z., TREUSSARD, J., WILLEN, P., (2007). The Theory of Life-Cycle Saving and Investing. [online] Federal Reserve Bank of Boston: Research Paper, no. 07-3. Available at: <http://ssrn.com/abstract=1002388> or <http://dx.doi.org/10.2139/ssrn.1002388> [Accessed 20 March 2013].

CARBONE, E., INFANTE, G., (2012). The Effect of a Short Planning Horizon on Intertemporal Consumption Choices. [online] LABSI: Research Paper. Available at: <http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2187911> [Accessed: 28 March 2013].

DUS, I., MAURER, R., MITCHELL, O. S., (2004). Betting on Death and Capital Markets in Retirement: A Shortfall Risk Analysis of Life Annuities versus Phased Withdrawal Plans. [online] Goethe-Universität, Frankfurt am Main and The Wharton School, University of Pennsylvania: Working Paper. Available at: <http://www.actuaries.org/AFIR/Colloquia/Boston/Dus_Maurer_Mitchell.pdf > [Accessed: 27 March 2013].

FELDMAN, L., PIETRZYK, R., ROKITA, P., (2014a). A practical method of determining longevity and premature-death risk aversion in households and some proposals of its application, in: Spiliopoulou, M., Schmidt-Thieme, L., Janning, R. (Eds.), Data Analysis, Machine Learning and Knowledge Discovery, Studies in Classification, Data Analysis, and Knowledge Organization. Springer International Publishing, pp. 255–264.

FELDMAN, L., PIETRZYK, R., ROKITA, P., (2014b). General strategies to meet household pension goal versus longevity risk, in: Lisowski J., Łyskawa K. (Eds.), Insurance in view of longevity/old age risk. Poznań University of Economics Publishing House, pp. 37–49.

GEYER, A., HANKE, M., WEISSENSTEINER, A., (2009). Life-cycle asset allocation and consumption using stochastic linear programming. The Journal of Computational Finance, 12(4), pp. 29–50.

GOMPERTZ, B., (1825). On the Nature of the Function Expressive of the Law of Human Mortality, and on a New Mode of Determining the Value of Life Contingencies. Philosophical Transactions of the Royal Society of London, 115, 513–585.

GONG, G., WEBB, A., (2008). Mortality Heterogeneity and the Distributional Consequences of Mandatory Annuitization. The Journal of Risk and Insurance, 75(4), pp. 1055–1079.

HUANG, H., MILEVSKY, M. A., (2011). Longevity Risk Aversion and Tax-Efficient Withdrawals. [online] SSRN. Available at: <http://ssrn.com/abstract=1961698> [Accessed: 22 March 2012].

HUANG, H., MILEVSKY, M. A., SALISBURY, T. S., (2012). Optimal Retirement Consumption with a Stochastic Force of Mortality. Insurance: Mathematics and Economics, 51(2), pp. 282–291.

KIRBY, K. N., HERRNSTEIN, R. J., (1995). Preference reversals due to myopic discounting of delayed reward. Psychological Science, 6(2), pp.83–89.

MAKEHAM, W. M., (1860). On the Law of Mortality and the Construction of Annuity Tables. Journal of the Institute of Actuaries and Assurance Magazine, 8, pp. 301–310.

MERTON, R. C., (1969). Lifetime portfolio selection under uncertainty: The continuous time case. The Review of Economics and Statistics, 51(3), pp. 247–257.

MERTON, R. C., (1971). Optimum consumption and portfolio rules in a continuous-time model. Journal of Economic Theory, 3(4), pp. 373–413.

MILEVSKY, M. A., HUANG, H., (2011). Spending Retirement on Planet Vulcan: The Impact of Longevity Risk Aversion on Optimal Withdrawal Rates. Financial Analysts Journal, 67(2), pp. 45–58.

MODIGLIANI, F., BRUMBERG, R. H., (1954). Utility analysis and the consumption function: an interpretation of cross-section data. In: Kenneth K. Kurihara, ed. 1954. Post-Keynesian Economics. New Brunswick, NJ: Rutgers University Press. pp. 388–436.

NISSENBAUM, M., RAASCH, B. J., RATNER, C., (2004). Ernst & Young's Personal Financial Planning Guide. John Wiley & Sons, Inc. 237–522.

RICHARD, S. F., (1975). Optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous time model. Journal of Financial Economics, 2, pp. 187–203.

SCHOLZ, K., SESHADRI, A., (2012). Health and Wealth In a Lifecycle Model. [online] University of Wisconsin-Madison and NBER and University of Wisconsin-Madison: Working Paper. Available at http://www.ssc.wisc.edu/~scholz/Research/Health_and_Wealth_v16.pdf [Accessed: 23 Nov. 2013].

SUNDARESAN, S., ZAPATERO, F., (1997). Valuation, optimal asset allocation and retirement incentives of pension plans. Review of Financial Studies, 10, pp. 631–660.

YAARI, M. E., (1965). Uncertain Lifetime, Life Insurance and Theory of the Consumer. The Review of Economic Studies, 32(2), pp.137–150.