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PROPOSITION OF STOCHASTIC POSTULATES FOR CHAIN INDICES

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ABSTRACT

This article presents and discusses a proposition of stochastic postulates for chain indices. The presented postulates are based on the assumption that prices and quantities are stochastic processes and we consider also the case when price processes are martingales. We define general conditions which allow the chain indices to satisfy these postulates.

Key words: chain indices, price index theory, stochastic processes, martingales.

1. Introduction

The idea of chain index construction, with weights changed every year, was probably first suggested by Alfred Marshall (1887). Marshall was concerned only with the practical problem of allowing for introduction of new commodities into an index of prices. He thought that the index would be greatly facilitated if weights were changed every year and the successive yearly indices linked or changed together by simple multiplication. Francois Divisia (1925) also postulated that the price index should depend not only on prices and quantities at considered moments $t = 0$ and $t = T$ but also on the movement of prices and quantities throughout the interval $[0, T]$. Divisia defined the index of prices by using a differential equation the solution of which was a curvilinear integral, and under assumptions that all functions $p_i(t)$ and $q_i(t)$, describing values of (respectively) prices and quantities of the considered N commodities ($i \in \{1, 2, ..., N\}$), exist at any point in time. Divisia's approach seems to be related to chain indices although it has a more general character (see Hulten (1973), Banerjee (1979)). Some authors treat Divisia's approach as some kind of justification for chain indices (see von der Lippe (2007). In fact, in some authors' opinion, all index formulas used in practice should approximate the Divisia index and chain indices should naturally translate the Divisia index into the reality of

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price observations at discrete points in time (see Feenstra, Reinsdorf (2000)). It should be added that there are infinitely many discrete approximations to the Divisia continuous time index because the value of this index depends on the path connecting moments $t=0$ and $t=T$ (see Samuelson and Swamy (1974), Vogt (1978)). The ideas of chain indices and Divisia's approach have many supporters and opponents (see Forsyth and Fowler (1981)) but we must admit with absolute certainty that chain indices play important role in practice (see for example Cho (2006)) and are recommended for deflation by the revised System of National Accounts[1] (see Von der Lippe (2001)). In this paper we try to supplement the theoretical background for chain indices by adding some new, stochastic postulates for them. According to NSA (News Stochastic Approach[2]) we treat the prices (and also quantities) of commodities as random variables. We claim that on the one hand these postulates are quite natural requirement but on the other hand some of them rules out known chain index formulas.

2. Chain indices and their properties

In the monograph of Von der Lippe (2007, p. 133) we can read: "*A chain index is essentially a specific type of aggregation (over intervals in time) and description of a time series rather than a comparison of two states taken in isolation; it provides a measure of the cumulated effect of successive steps (and the shape of the path) from 0 to 1, 1 to 2, …, $t-1$ to t* ". Let us denote by $P_{\tau-1,\tau}$ a *direct* price index formula (like Paasche, Laspeyres, Fisher or others). The chain index $\overline{P}_{0,t}$ calculated for the considered time interval $[0,t]$ can be expressed as a product of "links" $P_{\tau,\tau+1}$

$$\overline{P}_{0,t} = \prod_{\tau=0}^{t-1} P_{\tau,\tau+1} , \tag{1}$$

where each price index $P_{\tau,\tau+1}$ compares moment $\tau+1$ with the preceding moment τ .

In the literature we can meet a few major arguments for using chain indices in practice (Von der Lippe (2007)). We can list these arguments as follows: a) the "base" to which a time series of indices or of year-to-year growth rates refers is more relevant and realistic in the case of chain indices than in the case of traditional direct indices; b) some advantages are derived from a superior flexibility and adaptability as regards the structure of weights and the appearance

[1] According to the recommendations of System of National Accounts (SNA 1993) the chained Fisher price index should be used for both price level measurement and deflation (see also Von der Lippe (2007), p. 365).

[2] For more details see Clements and Izan (1987) and Selvanathan and Prasada Rao (1994).

of new and disappearance of old goods[1]; c) on the one hand chain indices are found unfavourable in the case of high price fluctuations or when cycles exist but on the other hand these indices are useful (in terms of desirable numerical results, like low inflation) "if individual prices and quantities tend to increase or decrease monotonically over time[2]"; d) in many authors' opinion the chain index version of various index formulas will yield less divergent results than the corresponding direct index version (see Von der Lippe (2007)); e) chain indices are recommended for deflation procedure by SNA. It is worth adding that many statements presenting advantages of chain indices over the direct indices focus on the links rather than the chain and they give arguments in favour of the chain index approach from the simple fact that the interval $[0,t]$ is subdivided into a number of sub-intervals and the chain index is derived from multiplications of links. Although in Europe chain indices are made mandatory for official statistics, not everyone shares the above-mentioned opinions and some criticism of the presented arguments can be found in Von der Lippe (2007). For instance, the Boskin Commission (1996) did not recommend chain indices but rather direct "superlative" indices (like Fisher formula), with weights from periods 0 and t.

However, we should also discuss axiomatic properties of chain indices to have a full list of arguments for or against these indices. It has to be mentioned here that from the axiomatic point of view the chain indices have many drawbacks. It is not only an easy theoretical possibility (see Von der Lippe (2007)) that chain indices may fail *the mean value test*[3], this has been shown empirically already at least once (Szulc (1983)). It means that a chain index may exceed the greatest individual price relative or can be smaller than the smallest price relative. The list of unsatisfied tests is longer – chain indices fail also *identity*, *monotonicity*, or *transitivity*. Many of arguments advanced to justify chain indices suffer from a lack of axiomatic tools to evaluate their properties. Note that only the link is an index in the sense of axiomatic approach. In fact, a chain is not and index and can violate many of tests despite all indices, playing the role of links, satisfy them all. Moreover, as it was above-mentioned in the introduction, chain indices depend on how an interval is subdivided. Hence, chain indices provide a summary description of a process rather than a comparison of two moments. In the next part of the paper we propose quite natural postulates for this process, where the last of them comes from finance. In our opinion the above-mentioned postulates can play an axiom role and we show its connection with the traditional *mean value property*.

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[1] The revised SNA 93 treats chain indices as „indices whose weighting structures are as up-to-date and relevant as possible". The SNA also found that chain indices make it "possible to obtain a much better match between products in consecutive time periods (…), given that products are continually disappearing from markets to be replaced by new product, or new qualities".

[2] SNA 93, para. 16.44.

[3] To read more about tests and axioms for price indices see Balk (1995).

3. New postulates and their interpretation

3.1. Stochastic model

Let us consider a group of N commodities. We observe them in discrete moments $\{t = 0,1,2,....,T\}$. Let us define a probability space $(\Omega, \Im, P)$. Let $F = \{\Im_t : t = 0,1,2,...,T\}$ be a filtration, i.e. each $\Im_t$ is an $\sigma-$algebra of Ω with $\Im_0 \subseteq \Im_s \subseteq \Im_t \subseteq \Im$ for any $s < t$. Without loss of generality, we assume $\Im_0 = \{\emptyset, \Omega\}$. The filtration F describes how the information about the market is revealed to the observer. We consider the following state-variables:

$p_i(t)$ - a price of the $i-$th commodity at time t,

$q_i(t)$ - a quantity of the $i-$th fund at time t,

$v_i(t) = p_i(t)q_i(t)$ - value of the $i-$th commodity at time t,

$v(t) = \sum_{i=1}^{N} v_i(t)$,

$v_i^*(t) = v_i(t)/v(t)$ - the percentage of a relative value of the $i-$th commodity at time t.

Here and subsequently, the symbol $X = Y$ means that the random variables X and Y are defined on $(\Omega, \Im, P)$ and it holds that $P(X = Y) = 1$. We assume that each $p_i(t)$ and $q_i(t)$ is adapted to $F = \{\Im_t : t = 0,1,2,...,T\}$, which means that each $p_i(t)$ and $q_i(t)$ is measurable with respect to $\Im_t$.

3.2. Stochastic postulates

As an initial stage of the discussion on postulates for chain indices we present the idea behind its definition. According to our best knowledge, the axiomatic price index theory is based on the deterministic approach and no test for indices is constructed for the case when prices and quantities are random. It would be quite interesting to rebuild the axioms on the stochastic case. For example, from the axiomatic approach (Balk (1995)) we know that one of the basic requirements for price indices is the so-called *proportionality,* which means that if all prices change $\lambda-$fold (from moment 0 to t) then the value of price index $P_{0,t}$ is also changed by λ. In other words, from

$$\frac{p_i(t)}{p_i(0)} = \lambda \text{, for } i = 1,2,....,N, \qquad (2)$$

we implicate $P_{0,t} = \lambda$.

The natural question is whether we should rebuild this axiom in stochastic case and require the following implication (let us call it *stochastic proportionality)*

$$E(\frac{p_i(t)}{p_i(0)}) = \lambda\,,\ i = 1,2,....,N \quad \Rightarrow \ E(P_{0,t}) = \lambda\,, \tag{3}$$

where $E(X)$ denotes the expected value of a random variable X .

Let us notice that in the special case, when $\lambda = 1$, we obtain the stochastic version of *identity* (constant prices test)

$$E(\frac{p_i(t)}{p_i(0)}) = 1,\ i = 1,2,....,N \quad \Rightarrow \ E(P_{0,t}) = 1. \tag{4}$$

On the basis of the implication (4) we construct the first postulate for chain indices:

Postulate 1

The chain index $\overline{P}_{0,t}$ should satisfy

$$E(\frac{p_i(\tau+1)}{p_i(\tau)}) = 1,\ i = 1,2,....,N\ ,\tau = 0,1,...,t-1 \Rightarrow \ E(\overline{P}_{0,t}) = 1. \tag{5}$$

As we know, for any random variables X and Y the condition $E(X) = E(Y)$ does not have to mean that $E(X/Y) = 1$. Thus, we propose another postulate, which on the one hand seems to be natural but on the other hand may be very restrictive:

Postulate 2

The chain index $\overline{P}_{0,t}$ should satisfy

$$E(p_i(\tau)) = p_i = const\,,\ i = 1,2,....,N\ ,\tau = 0,1,...,t \ \Rightarrow \ E(\overline{P}_{0,t}) = 1. \tag{6}$$

If we assume that prices of commodities are martingales (see Williams (1991)), which means that each $E\left|p_i(t)\right| < \infty$ and additionally

$$E(p_i(t)/\Im_s) = p_i(s),\ \ i = 1,2,....,N\,, \tag{7}$$

we get the following conditional expected value of the partial index

$$E(\frac{p_i(t)}{p_i(s)}/\Im_s) = \frac{1}{p_i(s)}E(p_i(t)/\Im_s) = \frac{p_i(s)}{p_i(s)} = 1. \tag{8}$$

For $s=0$ the equality (8) corresponds to the condition from the left side of the implication (4). Thus, we could expect that if the equality (7) holds for each price process, then the chain index should also behave like martingale and thus have the expected value constant in time. In other words, we form the following postulate for chain indices:

Postulate 3

If each process $\{p_i(t): t = 0,1,2,...,T\}$ is a $F-$martingale[1] for $i \in \{1,2,...,N\}$, then $\{\overline{P}_{0,t} : t = 0,1,2,...,T\}$ is also a $F-$martingale.

The postulate 3, although regarded as very important, seems to be less restrictive than postulates 1 and 2. In our opinion it has even axiomatic character because martingales have the expected value constant in time. Thus, the postulate 3 can play a role of a minimum requirement for chain indices. It is worth adding that the concept of martingale in probability theory is quite old and it was introduced by Paul Lévy in 1934, though he did not name it: the term *martingale* was introduced later by Ville (1939), who also extended the definition to continuous martingales. However, martingales play important role in modern probability, statistics and finance (Mansuy (2009)). In finance, in the case of measures of price dynamics on the given time interval, it is a very desirable property (see for example Gajek, Kałuszka (2000, 2001), Białek (2008)).

4. Some general remarks on the proposed postulates

In this section we discuss the general conditions for satisfying the presented postulates. In particular, we show some connections between traditional tests (postulates) for direct price indices and our postulates. We start our consideration from the theorem connected with the most fundamental postulate 3.

Theorem 1

If the direct price index formula (link) satisfies *the mean value test*[1] then the chain index $\overline{P}_{0,t}$, which is based on this link, satisfies the postulate 3.

[1] In probability theory, a martingale is a model of a fair game where knowledge of past events never helps predict the mean of the future winnings. In particular, a martingale is a sequence of random variables (i.e., a stochastic process) for which, at a particular time in the realized sequence, the expectation of the next value in the sequence is equal to the present observed value even given knowledge of all prior observed values at a current time. The assumption that prices are martingales is quite strong because it rules out any trends in relative prices. However, in this paper we discuss chain indices not only from the angle of official statistics but also from the angle of financial markets, where it is commonly considered assumption (see Samuelson (1965), Longstaff & Schwartz (2001), Mansuy (2009)). Moreover, we can use chain indices to construct measures of pension funds' efficiency (Białek (2012, 2013)), where the martingale pricing is one of the theoretical approaches (see for instance Gajek, Kałuszka (2001)).

Proof

We need to show that under the assumption that each process $\{p_i(t): t = 0,1,2,...,T\}$ is a $F-$martingale we would get for any moment t

$$E(\overline{P}_{0,t} / \Im_{t-1}) = \overline{P}_{0,t-1}. \tag{9}$$

Let us notice that from the fact that each $p_i(t)$ and $q_i(t)$ is measurable with respect to $\Im_t$ we conclude that also each stochastic process $P_{0,t}$ is measurable with respect to $\Im_t$. Thus we have

$$E(\overline{P}_{0,t} / \Im_{t-1}) = E(\prod_{\tau=0}^{t-1} P_{\tau,\tau+1} / \Im_{t-1}) = \prod_{\tau=0}^{t-2} P_{\tau,\tau+1} \cdot E(P_{t-1,t} / \Im_{t-1}). \tag{10}$$

From the assumption about *the mean value test* we have that

$$1 = E(\min_{i\in\{1,2,...,N\}} \frac{p_i(t)}{p_i(t-1)} / \Im_{t-1}) \leq E(P_{t-1,t} / \Im_{t-1}) \leq E(\max_{i\in\{1,2,...,N\}} \frac{p_i(t)}{p_i(t-1)} / \Im_{t-1}) = 1, \tag{11}$$

and hence

$$E(P_{t-1,t} / \Im_{t-1}) = 1. \tag{12}$$

From (10) and (12) we confirm (9), namely

$$E(\overline{P}_{0,t} / \Im_{t-1}) = E(\prod_{\tau=0}^{t-1} P_{\tau,\tau+1} / \Im_{t-1}) = \prod_{\tau=0}^{t-2} P_{\tau,\tau+1} = \overline{P}_{0,t-1}. \tag{13}$$

Remark 1

As it was mentioned above, the postulate 3 should be treated as a fundamental requirement. Let us notice that, by contraposition, if the chain index does not satisfy this postulate then the direct price index (link) does not fulfil *the mean value property*. It is worth adding that according to Pfouts (1966) *the mean value test* is one of the most essential properties of the index function. This fact is in conformity with our intuitive notion of an index to be a measure of a "representative" aggregated change. Moreover, *the mean value test* is included in systems of minimum requirements for price indices (see Eichhorn and Voeller, 1976). The immediate conclusion from the theorem 1 is that all used in practice price indices (like Laspeyres, Paasche, Fisher, Törnqvist, Walsh and other formulas – see Appendix) fulfil the postulate 1.

[1] The mean value test denotes that a value of the price index formula lies between minimum and maximum price relative. For instance, the Laspeyres price index can be expressed as follows: $P_{\tau,\tau+1}^{La} = \sum_{i=1}^{N} v_i^*(\tau) \frac{p_i(\tau+1)}{p_i(\tau)}$ and thus, being a convex combination of partial indices, this index fulfils the mean value property.

Remark 2

In Gajek and Kałuszka (2001) authors propose the stochastic definition of the average rate of return of a group of N open pension funds. Their measure is as follows: $R(0,T)=\prod_{\tau=0}^{T-1}(1+\sum_{i=1}^{N}v_i^*(\tau)r_i(\tau,\tau+1))-1,$ where $r_i(\tau,\tau+1)$ denotes the rate of return of $i-$th fund.

The major result of these authors is the theorem which allows one to state that $R(0,T)$ is martingale provided that unit prices are also martingales. It is easy to show (see Białek (2012)) that the measure of Gajek and Kałuszka can be expressed as a Laspeyres chain price index[1] and thus the links satisfy *the mean value test*. In other words, the thesis of the theorem by Gajek and Kałuszka is simply a consequence of the theorem 1.

Theorem 2

If any links $P_{s-1,s}$ and $P_{t-1,t}$ are independent (for $s\neq t$) and each link satisfies *the mean value test* then the chain index $\overline{P}_{0,t}$ fulfils the postulate 1.

Remark 3

The proof of the theorem 2 is quite obvious and it is omitted. The thesis of this theorem is a simple consequence of the known fact that independent links allow one to write

$$E(\overline{P}_{0,t})=E(\prod_{\tau=0}^{t-1}P_{\tau,\tau+1})=\prod_{\tau=0}^{t-1}E(P_{\tau,\tau+1}). \tag{14}$$

Remark 4

Let us notice that for any random variables X and Y we have (provided that the below expected values and standard deviations exist and $P(X=0)=0$)

$$\rho(X,\frac{Y}{X})=\frac{E(Y)-E(X)E(\frac{Y}{X})}{D(X)D(\frac{Y}{X})}, \tag{15}$$

where $\rho(X,Y/X)$ denotes the correlation coefficient between random variables X and Y/X and $D(X)$ denote the standard deviation of X. From (15) we get (if $E(X)\neq 0$)

[1] To read more about connections between measures of funds' efficiency and chain indices see Białek (2013).

$$E(\frac{Y}{X}) = \frac{E(Y)}{E(X)} - \frac{\rho(X, \frac{Y}{X})D(X)D(\frac{Y}{X})}{E(X)} = \frac{E(Y)}{E(X)} - \frac{\text{cov}(X, \frac{Y}{X})}{E(X)}, \qquad (16)$$

where $\text{cov}(X, \frac{Y}{X})$ denotes a covariance between random variables X and Y / X.

Thus, in the case of uncorrelated[1] X and Y / X, we obtain the equality

$$E(\frac{Y}{X}) = \frac{E(Y)}{E(X)}. \qquad (17)$$

The immediate conclusion is the following: if price processes fulfil[2]

$$\text{cov}(p_i(\tau), \frac{p_i(\tau+1)}{p_i(\tau)}) = 0, \qquad (18)$$

then

$$E(\frac{p_i(\tau+1)}{p_i(\tau)}) = \frac{E(p_i(\tau+1))}{E(p_i(\tau))}. \qquad (19)$$

In other words we have the equivalence for each $i = 1,2,...., N$

$$\forall\ \tau = 0,1,...,t-1\ \ E(\frac{p_i(\tau+1)}{p_i(\tau)}) = 1 \Leftrightarrow \forall\ \tau = 0,1,...,t\ \ E(p_i(\tau)) = const \qquad (20)$$

and it leads to the final conclusion that if the condition (18) holds then the postulates 1 and 2 are equivalent. Moreover, we can formulate the following theorem:

Theorem 3

If the direct price index formula (link) satisfies *the mean value test and the circular test*[3] and, moreover, $p_i(0)$ and $p_i(t)/p_i(0)$ are uncorrelated for each $i \in \{1,2,..., N\}$, then the chain index $\overline{P}_{0,t}$, which is based on this link, satisfies the postulate 2.

Proof

Let us assume, according to the assumptions from the postulate 2, that

$$E(p_i(\tau)) = p_i = const,\ i = 1,2,...., N\ \ , \tau = 0,1,...,t. \qquad (21)$$

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[1] For instance, such a theoretical situation was considered in Frishman (1971).

[2] It can be quite natural assumption because it requires that prices and relative price changes are uncorrelated.

[3] The circular test denotes that for any moments $s < t < v$ it holds that $P_{s,v} = P_{s,t} P_{t,v}$. The circularity is one of the most restrictive tests in price index theory but often considered in theoretical papers and monographs (see Balk (1995), von der Lippe (2007)).

The satisfied *circular test* leads to the following equality

$$E(\overline{P}_{0,t}) = E(\prod_{\tau=0}^{t-1} P_{\tau,\tau+1}) = E(P_{0,1} \cdot P_{1,2} \cdot ... \cdot P_{t-1,t}) = E(P_{0,t}). \qquad (22)$$

The satisfied *the mean value test* leads to the following relation

$$E(\frac{p_n(t)}{p_n(0)}) = E(\min_{i\in\{1,2,...,N\}} \frac{p_i(t)}{p_i(0)}) \le E(P_{0,t}) \le E(\max_{i\in\{1,2,...,N\}} \frac{p_i(t)}{p_i(0)}) = E(\frac{p_m(t)}{p_m(0)}), \qquad (23)$$

for some $n, m \in \{1,2,..., N\}$.

From the fact that $\mathrm{cov}(p_n(0), \frac{p_n(t)}{p_n(0)}) = 0$ and $\mathrm{cov}(p_m(0), \frac{p_m(t)}{p_m(0)}) = 0$ we conclude that

$$E(\frac{p_n(t)}{p_n(0)}) = \frac{E(p_n(t))}{E(p_n(0))} = \frac{p_i}{p_i} = 1, \qquad (24)$$

and analogically

$$E(\frac{p_m(t)}{p_m(0)}) = \frac{E(p_m(t))}{E(p_m(0))} = \frac{p_i}{p_i} = 1. \qquad (25)$$

From (23), (24) and (25) we obtain $E(P_{0,t}) = 1$, which confirms that the postulate 2 is fulfilled.

5. Conclusions

In the paper three stochastic postulates for chain indices are proposed, as an alternative for the classic axiomatic price index theory. The novelty of the presented approach is due to treating the prices and quantities as stochastic processes. The presented postulates have different nature – the postulates 1 and 2 are quite restrictive and we can treat them as some desirable properties but the postulate 3, connected with *the mean value property*, has axiomatic character. Under some additional condition the postulates 1 and 2 are equivalent (see Remark 4). If these postulates are not equivalent we can still show conditions which allow one to fulfil each of the postulates. The most restrictive assumption is in the theorem 3 because it requires *the circularity*. However, there are price index formulas satisfying *the circular test,* like the Walsh price index (see Von der Lippe (2007)). This discussion serves also as a kind of introduction to the author's future research agenda on chain index theory. In our opinion the theorems 1, 2 and 3 are a good starting point because all consideration begins from the basic *proportionality* and *identity*.

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Appendix 1.

For example, according to the thesis of the theorem 1, the following direct price indices (links) guarantee that postulate 1 is satisfied:

- the Laspeyres price index

$$P_{\tau,\tau+1}^{La} = \frac{\sum_{i=1}^{N} q_i(\tau) p_i(\tau+1)}{\sum_{i=1}^{N} q_i(\tau) p_i(\tau)};$$

- the logarithmic Laspeyres price index

$$P_{\tau,\tau+1}^{LLa} = \prod_{i=1}^{N} \left(\frac{p_i(\tau+1)}{p_i(\tau)}\right)^{v_i^*(\tau)};$$

- the Paasche price index

$$P_{\tau,\tau+1}^{Pa} = \frac{\sum_{i=1}^{N} q_i(\tau+1) p_i(\tau+1)}{\sum_{i=1}^{N} q_i(\tau+1) p_i(\tau)};$$

- the logarithmic Paasche price index

$$P_{\tau,\tau+1}^{LPa} = \prod_{i=1}^{N} \left(\frac{p_i(\tau+1)}{p_i(\tau)}\right)^{v_i^*(\tau+1)};$$

- the Törnqvist price index

$$P_{\tau,\tau+1}^{T} = \prod_{i=1}^{N} \left(\frac{p_i(\tau+1)}{p_i(\tau)}\right)^{\frac{1}{2}(v_i^*(\tau)+v_i^*(\tau+1))};$$

- the Fisher price index

$$P_{\tau,\tau+1}^{F} = \sqrt{P_{\tau,\tau+1}^{La} P_{\tau,\tau+1}^{Pa}};$$

- the Walsh price index

$$P_{\tau,\tau+1}^{W} = \frac{\sum_{i=1}^{N} p_i(\tau+1)\sqrt{q_i(\tau) q_i(\tau+1)}}{\sum_{i=1}^{N} p_i(\tau)\sqrt{q_i(\tau) q_i(\tau+1)}}.$$