STATISTICS IN TRANSITION new series, March 2018
Vol. 19, No. 1, pp. 25–44, DOI 10.21307/stattrans-2018-002

IMPROVED ROTATION PATTERNS USING TWO AUXILIARY VARIABLES IN SUCCESSIVE SAMPLING

Jaishree Prabha Karna[1]**, Dilip Chandra Nath**

ABSTRACT

The present paper emphasizes the role of two auxiliary variables on both the occasions to improve the precision of estimates at the current (second) occasion in two-occasion successive sampling. Information on two auxiliary variables, which are positively correlated with the study variable, has been used with the aid of exponential type structures and an efficient estimation procedure of population mean on the current (second) occasion has been suggested. The behaviour of the proposed estimator has been studied and compared with the sample mean estimator, when there is no matching from the previous occasion and natural successive sampling estimator, which is a linear combination of the means of the matched and unmatched portions of the sample at the current (second) occasion. Optimal replacement strategy is also discussed. The concluding remarks are discussed justifying utility of the proposed sampling scheme. The results have been well supported analytically as well as empirically by using real life data.

Key words: exponential type estimators, bias, mean squared error, optimum replacement strategy.
Mathematics subject classification: 62D05

1. Introduction

Repeated surveys over years or seasons or months are commonly used on many occasions for estimating same characteristics at different points of time. The information collected on previous occasion can be used to study the change or the total value over occasion for the character and also in addition to study the average value for the most recent occasion. There are several possibilities: (i) the same sample may be used on each occasion (ii) a new sample may be drawn on each occasion or (iii) a part of the sample may be retained while the remainder of the sample may be drawn afresh. Intuition suggests that for estimating changes from one occasion to the next, it may be best to retain the same sample on each occasion, while for estimating the mean on each occasion it may be advised to draw a fresh sample on each occasion. If it is desired to estimate the population mean on each occasion and also the change from one occasion to the next, it is

[1] Department of Statistics, Gauhati University, Guwahati – 781014, India.
E-mail: jaishree.prabha@gmail.com

always better to retain a part of the sample and draw the remainder of the sample afresh.

In successive (rotation) sampling, it is more advantageous to utilize the entire information collected in the previous investigations (occasions), to cite one may refer the papers by Jessen (1942), Patterson (1950), Rao and Graham (1964), Gupta (1979), Das (1982) and Chaturvedi and Tripathi (1983). Sen (1971) has also used this technique successfully with the utilization of information on two auxiliary variables, which was readily available on previous occasion, and proposed the estimators of population mean on the current occasion in two-occasion successive sampling. Sen (1972, 1973) further extended his work for several auxiliary variables. In many situations, information on an auxiliary variable may be readily available on the first as well as on the second occasion. For instance, to study the case of public health and welfare of a state or country, several factors are available that can be treated as auxiliary variables, such as the number of beds in different hospitals may be known, number of doctors and supporting staffs may be available, the amount of funds available for medicine etc. may be known. Likewise, there is a wealth of information available, which if used efficiently can improve the precision of estimates. Utilizing the auxiliary information on both the occasions Feng and Zou (1997), Biradar and Singh (2001), Singh (2005) and Singh and Karna (2009 a, b) proposed ratio and regression type estimators for estimating the population mean on the current (second) occasion in two-occasion successive sampling. More recently, the contributions of Ralte and Das (2015), Singh and Pal (2015), Karna and Nath (2016) and Beevi and Chandran (2017) established beneficial results by using the auxiliary variables on both the occasions.

Exponential type estimators support increasing the precision of the estimates of population parameters such as mean, median, total, etc. The exponential ratio and product type estimators in the estimation of finite population mean was introduced by Bahl and Tuteja (1991), when variable of interest and auxiliary variable is negatively or positively correlated. Further, the works of Upadhyaya et al. (2011), Yadav and Cadilar (2013), Singh and Pal (2017) examine the advantageous property of the exponential type estimators.

In line with the preceding works, we propose more an effective and relevant estimator using exponential type estimators for population mean at the current occasion in two-occasion successive sampling. Properties of the proposed estimator and optimum replacement policy have been discussed. Empirical support have been given to validate the theoretical results.

2. Formulation of the estimator Δ

Let U = (U_1, U_2, …, U_N) be the finite population of N units, which has been sampled over two occasions. The character under study is denoted by x (y) on the first (second) occasion respectively. Assume that the information on two auxiliary variables z_1 and z_2, whose population means $\bar{Z}_1$ and $\bar{Z}_2$ are known, is available on the current (second) occasion and is closely related (positively correlated) to y on the second occasion. Let a simple random sample (without replacement) of size n be selected on the first occasion. A random sub-sample of m = nλ units is retained (matched) for its use on the second occasion, while a fresh simple random sample (without replacement) of u = (n-m) = nμ units is drawn on the second occasion from the entire population so that the sample size on the second occasion is also n. λ and μ (λ + μ = 1) are the fractions of matched and fresh samples at the second (current) occasion. Further, we consider the following notations throughout this work:

$\bar{X}, \bar{Y}, \bar{Z}_1, \bar{Z}_2$: population means of the variables x, y, z_1 and z_2 respectively;

$\bar{x}_n, \bar{x}_m, \bar{y}_u, \bar{y}_m, \bar{z}_{1u}, \bar{z}_{2u}, \bar{z}_{1n}, \bar{z}_{2n}$: sample means of the respective variables based on the sample sizes shown in suffices;

b_{yx}: sample regression coefficient of the variable y on x;

β_{yx} : population regression coefficient of the variable y on x;

ρ_{yx}, ρ_{yz1} , ρ_{yz2}, ρ_{xz1}, ρ_{xz2}, ρ_{z1z2} : correlation coefficients between the variables shown in suffices;

$S_X^2 = (N-1)^{-1}\sum_{i=1}^{N}(x_i - \bar{X})^2$: population mean square of the variable x;

$S_X^2, S_Y^2, S_{Z1}^2, S_{Z2}^2$: population mean squares of the variables x, y, z_1 and z_2 respectively.

Utilizing information on two auxiliary variables, two different estimators of the population mean $\bar{Y}$ on the current (second) occasion may be considered. Motivated with the work of Bahl and Tuteja (1991), the estimator based on the fresh sample of size u drawn on the second occasion is defined by

$$\Delta_u = \bar{y}_u \exp\left(\frac{\bar{Z}_1 - \bar{z}_{1u}}{\bar{Z}_1 + \bar{z}_{1u}}\right) \exp\left(\frac{\bar{Z}_2 - \bar{z}_{2u}}{\bar{Z}_2 + \bar{z}_{2u}}\right). \quad (1)$$

The second estimator Δ_m is also a modified chain ratio type estimator based on the sample of size m, common with both the occasions, and is defined as

$$\Delta_m = \left[\bar{y}_m + b_{yx}(m)(\bar{x}_n - \bar{x}_m)\right] \exp\left(\frac{\bar{Z}_1 - \bar{z}_{1n}}{\bar{Z}_1 + \bar{z}_{1n}}\right) \exp\left(\frac{\bar{Z}_2 - \bar{z}_{2n}}{\bar{Z}_2 + \bar{z}_{2n}}\right). \quad (2)$$

Now, considering the convex linear combination of Δ_u and Δ_m , we define the final estimator of population mean $\bar{Y}$ on the current occasion as

$$\Delta = \varphi\Delta_u + (1-\varphi)\Delta_m \quad (3)$$

where φ is an unknown constant to be determined so as to minimize mean square error of the estimator Δ.

For estimating the mean on each occasion the estimator Δ_u is suitable, which implies that more belief on Δ_u could be shown by choosing φ as 1 (or close to 1), while for estimating the change from one occasion to the next, the estimator Δ_m could be more useful so φ might be chosen as 0 (or close to 0). For asserting both the problems simultaneously, a suitable (optimum) choice of φ is required.

3. Properties of the proposed estimator Δ

3.1 Bias and mean square error of estimator Δ

Since, Δ_u and Δ_m are ratio and chain-type regression in ratio estimators respectively, they are biased for population mean $\bar{Y}$. Therefore, the resulting estimators Δ defined in (3) is also biased estimator of $\bar{Y}$. To obtain bias B (.) and mean square errors M (.) of Δ up to the first order of approximations we consider the following transformations:

$\bar{y}_u = (1+e_1)\bar{Y}$, $\bar{y}_m = (1+e_2)\bar{Y}$, $\bar{x}_m = (1+e_3)\bar{X}$, $\bar{x}_n = (1+e_4)\bar{X}$,
$\bar{z}_{1u} = (1+e_5)\bar{Z}_1$, $\bar{z}_{1n} = (1+e_6)\bar{Z}_1$, $\bar{z}_{2u} = (1+e_7)\bar{Z}_2$, $\bar{z}_{2n} = (1+e_8)\bar{Z}_2$,
$s_{yx}(m) = (1+e_9)S_{yx}$, $s_x^2(m) = (1+e_{10})S_x^2$; such that $E(e_k) = 0$ and
$|e_k| < 1 \ \forall \ k = 1, 2, 3, ..., 10.$

Relative variances and covariances are derived as

$$E(e_1^2) = \left(\frac{1}{u} - \frac{1}{N}\right)C_y^2, \quad E(e_2^2) = \left(\frac{1}{m} - \frac{1}{N}\right)C_y^2,$$

$$E(e_3^2) = \left(\frac{1}{m} - \frac{1}{N}\right)C_x^2, \quad E(e_4^2) = \left(\frac{1}{n} - \frac{1}{N}\right)C_x^2,$$

$$E(e_5^2) = \left(\frac{1}{u} - \frac{1}{N}\right)C_{z1}^2, \quad E(e_6^2) = \left(\frac{1}{n} - \frac{1}{N}\right)C_{z1}^2,$$

$$E(e_7^2) = \left(\frac{1}{u} - \frac{1}{N}\right)C_{z2}^2, \quad E(e_8^2) = \left(\frac{1}{n} - \frac{1}{N}\right)C_{z2}^2,$$

$$E(e_3e_4) = \left(\frac{1}{n} - \frac{1}{N}\right)C_x^2, \quad E(e_1e_5) = \left(\frac{1}{u} - \frac{1}{N}\right)\rho_{yz1}C_yC_{z1},$$

$$E(e_1e_7) = \left(\frac{1}{u} - \frac{1}{N}\right)\rho_{yz2}C_yC_{z2}, \quad E(e_5e_7) = \left(\frac{1}{u} - \frac{1}{N}\right)\rho_{z1z2}C_{z1}C_{z2},$$

$$E(e_2e_6) = \left(\frac{1}{n} - \frac{1}{N}\right)\rho_{yz1}C_yC_{z1}, \quad E(e_2e_8) = \left(\frac{1}{n} - \frac{1}{N}\right)\rho_{yz2}C_yC_{z2},$$

$$E(e_6e_8) = \left(\frac{1}{n}-\frac{1}{N}\right)\rho_{z1z2}C_{z1}C_{z2}, \quad E(e_2e_3) = \left(\frac{1}{m}-\frac{1}{N}\right)\rho_{yx}C_yC_x,$$

$$E(e_2e_4) = \left(\frac{1}{n}-\frac{1}{N}\right)\rho_{yx}C_yC_x, \quad E(e_3e_6) = \left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xz1}C_xC_{z1},$$

$$E(e_4e_6) = \left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xz1}C_xC_{z1}, \quad E(e_3e_8) = \left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xz2}C_xC_{z2},$$

$$E(e_4e_8) = \left(\frac{1}{n}-\frac{1}{N}\right)\rho_{xz2}C_xC_{z2}, \quad E(e_3e_9) = \left(\frac{1}{m}-\frac{1}{N}\right)\frac{\alpha_{2100}}{\bar{X}S_{yx}},$$

$$E(e_4e_9) = \left(\frac{1}{n}-\frac{1}{N}\right)\frac{\alpha_{2100}}{\bar{X}S_{yx}}, \quad E(e_3e_{10}) = \left(\frac{1}{m}-\frac{1}{N}\right)\frac{\alpha_{3000}}{\bar{X}S_x^2},$$

$$E(e_4e_{10}) = \left(\frac{1}{n}-\frac{1}{N}\right)\frac{\alpha_{3000}}{\bar{X}S_x^2}.$$

where $\alpha_{qrst} = E\left[\left(x-\bar{X}\right)^q\left(y-\bar{Y}\right)^r\left(z_1-\bar{Z}_1\right)^s\left(z_2-\bar{Z}_2\right)^t\right]$; ((q, r, s, t) $\geq$ 0 are integers)

Under the above transformations Δ_u and Δ_m take the following forms:

$$\Delta_u = (1+e_1)\bar{Y}\exp\left(\frac{-e_5}{2+e_5}\right)\exp\left(\frac{-e_7}{2+e_7}\right) \tag{4}$$

$$\Delta_m = \left[(1+e_2)\bar{Y}+\left(1+e_9\right)\left(e_4-e_3\right)\left(1+e_{10}\right)^{-1}\beta_{yx}\bar{X}\right]\exp\left(\frac{-e_6}{2+e_6}\right)\exp\left(\frac{-e_8}{2+e_8}\right). \tag{5}$$

Subsequently, we have the following theorems:

Theorem 3.1: Bias of the estimator Δ to the first order of approximations is obtained as

$$B(T) = \varphi\, B(\Delta_u) + (1-\varphi)\, B(\Delta_m) \tag{6}$$

where

$$B\left(\Delta_u\right) = \frac{1}{\bar{Y}}\left(\frac{1}{u}-\frac{1}{N}\right)\left(\frac{3}{4}-\frac{\rho_{yz2}}{2}-\frac{\rho_{yz1}}{2}+\frac{\rho_{z1z2}}{4}\right)S_y^2 \tag{7}$$

$$B(\Delta_m) = \frac{1}{\bar{Y}}\left(\frac{1}{n}-\frac{1}{N}\right)\left(\frac{3}{4}-\frac{\rho_{yz2}}{2}-\frac{\rho_{yz1}}{2}+\frac{\rho_{z1z2}}{4}\right)+\left(\frac{1}{m}-\frac{1}{n}\right)\left(\frac{\alpha_{2100}}{S_x^2}+\frac{\alpha_{3000}S_{yx}}{S_x^4}\right)$$
$$-\frac{1}{2}\left(\frac{1}{n}-\frac{1}{N}\right)\frac{\rho_{xz2}S_{yx}}{\bar{X}} \qquad (8)$$

Proof: The bias of the estimator Δ is given by

$$B(\Delta) = E\left[\Delta-\bar{Y}\right] = \varphi E(\Delta_u-\bar{Y}) + (1-\varphi)\, E(\Delta_m-\bar{Y})$$
$$= \varphi B(\Delta_u)+\left(1-\varphi\right)B(\Delta_m) \qquad (9)$$

where $B(\Delta_u) = E\left[\Delta_u-\bar{Y}\right]$ and $B(\Delta_m) = E\left[\Delta_m-\bar{Y}\right]$.

The bias of Δ_u and Δ_m is derived as follows:

$$B(\Delta_u) = E\left[\Delta_u-\bar{Y}\right] .$$

Substituting the expression of Δ_u from from equation (4), we get

$$B(\Delta_u) = E\left[(1+e_1)\bar{Y}\exp\left(\frac{-e_5}{2+e_5}\right)\exp\left(\frac{-e_7}{2+e_7}\right)-\bar{Y}\right] .$$

Now, expanding the right hand side of the above expression, taking expectations and retaining the terms up to the first order of approximations, we have

$$B(\Delta_u) = \frac{1}{\bar{Y}}\left(\frac{1}{u}-\frac{1}{N}\right)\left(\frac{3}{4}-\frac{\rho_{yz2}}{2}-\frac{\rho_{yz1}}{2}+\frac{\rho_{z1z2}}{4}\right). \qquad (10)$$

Similarly

$$B(\Delta_m) = E\left[\Delta_m-\bar{Y}\right]$$
$$= E\left[\left\{(1+e_2)\bar{Y}+(1+e_9)(e_4-e_3)(1+e_{10})^{-1}\beta_{yx}\bar{X}\right\}\exp\left(\frac{-e_6}{2+e_6}\right)\exp\left(\frac{-e_8}{2+e_8}\right)-\bar{Y}\right]$$
$$= \frac{1}{\bar{Y}}\left(\frac{1}{n}-\frac{1}{N}\right)\left(\frac{3}{4}-\frac{\rho_{yz2}}{2}-\frac{\rho_{yz1}}{2}+\frac{\rho_{z1z2}}{4}\right)+\left(\frac{1}{m}-\frac{1}{n}\right)\left(\frac{\alpha_{2100}}{S_x^2}+\frac{\alpha_{3000}S_{yx}}{S_x^4}\right)$$
$$-\frac{1}{2}\left(\frac{1}{n}-\frac{1}{N}\right)\frac{\rho_{xz2}S_{yx}}{\bar{X}} . \qquad (11)$$

Substituting the values of B(Δ$_u$) and B(Δ$_m$) from equations (10) and (11) in the equation (9) we get the bias of estimator Δ as shown in equation (6).

Theorem 3.2: Mean square error of the estimator Δ to the first order of approximations is obtained as

$$M(\Delta) = \varphi^2 M(\Delta_u) + (1-\varphi)^2 M(\Delta_m) + 2\varphi(1-\varphi)\mathrm{Cov}(\Delta_u, \Delta_m) \quad (12)$$

where

$$M(\Delta_u) = E(\Delta_u - \bar{Y})^2 = \left(\frac{1}{u} - \frac{1}{N}\right) A_1 S_y^2 \quad (13)$$

$$M(\Delta_m) = E(\Delta_m - \bar{Y})^2 = \left[\left(\frac{1}{m}\right) A_2 + \left(\frac{1}{n}\right) A_3 - \left(\frac{1}{N}\right) A_4\right] S_y^2 \quad (14)$$

and $\mathrm{Cov}(\Delta_u, \Delta_m) = E\left[(\Delta_u - \bar{Y})(\Delta_m - \bar{Y})\right] = -\frac{A_1 S_y^2}{N}$ (15)

where $A_1 = \frac{3}{2} + \frac{1}{2}\left\{\rho_{z1z2} - 2\left(\rho_{yz2} + \rho_{yz1}\right)\right\}$, $A_2 = 1 - \rho_{yx}^2$,

$A_3 = \frac{1}{2}\left\{\rho_{z1z2} - 2\left(\rho_{yz2} + \rho_{yz1}\right)\right\} + \rho_{yx}^2$ and $A_4 = \frac{1}{2}\left\{\rho_{z1z2} - 2\left(\rho_{yz2} + \rho_{yz1}\right)\right\} + 1.$

Proof: It is obvious that mean square error of the estimator Δ is given by

$$M(\Delta) = E\left[\Delta - \bar{Y}\right]^2 = E\left[\varphi(\Delta_u - \bar{Y}) + (1-\varphi)(\Delta_m - \bar{Y})\right]^2$$

$$= \varphi^2 M(\Delta_u) + (1-\varphi)^2 M(\Delta_m) + 2\varphi(1-\varphi)\mathrm{Cov}(\Delta_u, \Delta_m) \quad (16)$$

where $M(\Delta_u) = E\left[\Delta_u - \bar{Y}\right]^2$, $M(\Delta_m) = E\left[\Delta_m - \bar{Y}\right]^2$ and

$\mathrm{Cov}(\Delta_u, \Delta_m) = E\left[(\Delta_u - \bar{Y})(\Delta_m - \bar{Y})\right]$.

The mean square errors of Δ$_u$ and Δ$_m$ are derived as follows:

$$M(\Delta_u) = E\left[\Delta_u - \bar{Y}\right]^2 .$$

Substituting the expression of Δ$_u$ from equation (4), we get

$$M(\Delta_u) = E\left[(1+e_1)\bar{Y}\exp\left(\frac{-e_5}{2+e_5}\right)\exp\left(\frac{-e_7}{2+e_7}\right) - \bar{Y}\right]^2 .$$

Now, expanding the right-hand side of above expression, taking expectations and retaining the terms up to the first order of approximations, we have

$$M(\Delta_u) = E\left(\Delta_u - \bar{Y}\right)^2 = \left(\frac{1}{u} - \frac{1}{N}\right)\left[\frac{3}{2} + \frac{1}{2}\left\{\rho_{z1z2} - 2\left(\rho_{yz2} + \rho_{yz1}\right)\right\}\right] S_y^2 . \tag{17}$$

Similarly

$$M\left(\Delta_m\right) = E\left[\Delta_m - \bar{Y}\right]^2$$

$$= E\left[\left\{(1+e_2)\bar{Y} + \left(1+e_9\right)\left(e_4 - e_3\right)\left(1+e_{10}\right)^{-1}\beta_{yx}\bar{X}\right\}\exp\left(\frac{-e_6}{2+e_6}\right)\exp\left(\frac{-e_8}{2+e_8}\right) - \bar{Y}\right]^2$$

$$= \left[\left(\frac{1}{m}\right)\left(1-\rho_{yx}^2\right) + \left(\frac{1}{n}\right)\left\{\frac{1}{2}\left\{\rho_{z1z2} - 2\left(\rho_{yz2} + \rho_{yz1}\right)\right\} + \rho_{yx}^2\right\} - \left(\frac{1}{N}\right)\left\{\frac{1}{2}\left\{\rho_{z1z2} - 2\left(\rho_{yz2} + \rho_{yz1}\right)\right\} + 1\right\}\right] S_y^2$$

(18)

and

$$Cov\left(\Delta_u, \Delta_m\right) = E\left[\left(\Delta_u - \bar{Y}\right)\left(\Delta_m - \bar{Y}\right)\right] = -\left[\frac{3}{2} + \frac{1}{2}\left\{\rho_{z1z2} - 2\left(\rho_{yz2} + \rho_{yz1}\right)\right\}\right]\frac{S_y^2}{N} . \tag{19}$$

Substituting the values of $M(\Delta_u)$, $M(\Delta_m)$ and $Cov(\Delta_u, \Delta_m)$ from equations (17), (18) and (19) in the equation (16) we get mean square error of estimator Δ as shown in equation (12).

Remark 1: Since x and y are the same study variable over two occasions and z_1 and z_2 are auxiliary variables positively correlated with x and y, therefore, considering the stable behaviour of coefficient of variations [Reddy (1978)] the expressions of bias mean square errors in equations (6) and (12) are derived under the assumption that the coefficients of variation of x, y and z_1 and z_2 are approximately equal, i.e. $C_x = C_y = C_{z1} = C_{z2}$.

3.2. Minimum mean square error of Δ

The mean square error of the estimator Δ in equation (12) is the function of unknown constant φ, therefore, it is minimized with respect to φ and subsequently the optimum value of φ is obtained as

$$\varphi_{opt} = \frac{M(\Delta_m) - Cov\left(\Delta_u, \Delta_m\right)}{M(\Delta_u) + M(\Delta_m) - 2Cov\left(\Delta_u, \Delta_m\right)} . \tag{20}$$

Now, substituting the value of φ_{opt} in equation (12), we get the optimum mean square error of Δ as

$$M(\Delta)_{opt} = \frac{M(\Delta_u).M(\Delta_m)-\left\{Cov\left(\Delta_u, \Delta_m\right)\right\}^2}{M(\Delta_u)+M(\Delta_m)-2Cov\left(\Delta_u, \Delta_m\right)} . \tag{21}$$

Further, substituting the values from equations (17) - (19) in equations (20) and (21), we get the simplified value of M $(\Delta)_{opt}$ as:

$$M(\Delta)_{opt} = \left[\frac{P_1+\mu P_2+\mu^2 P_3}{A_1+\mu P_4+\mu^2 P_5}\right]\frac{S_y^2}{n} \tag{22}$$

where $P_1 = A_1\left(A_2+A_3-fA_4\right)$, $P_2 = A_1\left\{f^2\left(A_4-A_1\right)+f\left(A_4-A_3-A_2\right)-A_3\right\}$,
$P_3 = A_1\left\{f^2\left(A_1-A_4\right)+fA_3\right\}$, $P_4 = f\left(A_1-A_4\right)+\left(A_3+A_2-A_1\right)$,
$P_5 = \left\{f\left(A_4-A_1\right)-A_3\right\}$.

4. Optimum replacement policy

To determine the optimum value of μ (fraction of sample to be drawn afresh on the current occasion) so that population mean $\bar{Y}$ may be estimated with maximum precision and at the lowest cost, we minimize the mean square error of Δ given in equation (22) with respect to μ , which results in a quadratic equation in μ. The quadratic equation in μ and respective solutions of μ say $\hat{\mu}$ are given below:

$$Q_1\mu^2 + 2\mu Q_2+Q_3 = 0 \tag{23}$$

$$\hat{\mu} = \frac{-Q_2 \pm\sqrt{Q_2^2-Q_1Q_3}}{Q_1} \tag{24}$$

where $Q_1 = P_3P_4 - P_2P_5$, $Q_2 = A_1P_3 - P_1P_5$, $Q_3 = A_1P_2 - P_1P_4$.

From equation (24), it is clear that the real values of $\hat{\mu}$ exist if the quantity under square root is greater than or equal to zero. For any combinations of ρ_{yx} and ρ_{yz} , which satisfy the condition of real solutions, two real values of $\hat{\mu}$ are possible. Hence, while choosing the value of $\hat{\mu}$, it should be remembered that $0 \le \hat{\mu} \le 1$, all others value of $\hat{\mu}$ are inadmissible. If both the values are admissible the lowest one will be the best choice because it reduces the cost of the survey. Substituting the admissible value of $\hat{\mu}$ say μ^* from equation (24) into equation

(22), we have the optimum value of mean square error of the estimator Δ, which is shown below

$$M(T)_{opt^*} = \left[\frac{P_1+\mu^* P_2+\mu^{*2} P_3}{A_1+\mu^* P_4+\mu^{*2} P_5} \right] \frac{S_y^2}{n}. \tag{25}$$

5. Efficiency comparison

For comparing the efficiencies of the proposed estimator we have considered two estimators: (i) sample mean estimator $\bar{y}_n$, when there is no matching and (ii) natural successive sampling estimator $\hat{\bar{Y}} = \varphi^* \bar{y}_u + (1-\varphi^*)\bar{y}_m^{'}$, when no auxiliary information is used at any occasion, where $\bar{y}_m^{'} = \bar{y}_m + \beta_{yx}\left(\bar{x}_n - \bar{x}_m\right)$.

Clearly different estimators proposed in successive (rotation) sampling have their own assumptions and limitations; therefore, practically it is not feasible to compare the proposed estimators with the other estimators available in the survey literature. Hence, the efficiency comparisons have been made with the sample mean estimator and the natural successive sampling estimator. The percent relative efficiencies of the estimator Δ with respect to $\bar{y}_n$ and $\hat{\bar{Y}}$, have been obtained for different choices of correlations. Since $\bar{y}_n$ and $\hat{\bar{Y}}$ are unbiased estimators of $\bar{Y}$, therefore, following Sukhatme et al. (1984) the variance of $\bar{y}_n$ and optimum variance of $\hat{\bar{Y}}$ are given by

$$V(\bar{y}_n) = \left(\frac{1}{n} - \frac{1}{N}\right) S_y^2 \tag{26}$$

$$V\left(\hat{\bar{Y}}\right)_{opt^*} = \left[1+\sqrt{1-\rho_{yx}^2}\right] \frac{S_y^2}{2n} - \frac{S_y^2}{N}. \tag{27}$$

For N = 5000, n = 500 and different choices of ρ_{z1z2}, ρ_{yz1}, ρ_{yz2} and ρ_{yx} Tables 1-3 give the optimum values of μ and percent relative efficiencies E_1 and E_2 of Δ with respect to $\bar{y}_n$ and $\hat{\bar{Y}}$ respectively, where

$$E_1 = \frac{V(\bar{y}_n)}{M(\Delta)_{opt^*}} \times 100 \text{ and } E_2 = \frac{V\left(\hat{\bar{Y}}\right)_{opt^*}}{M(\Delta)_{opt^*}} \times 100.$$

Remark 2: To compare the performance of the estimators Δ with respect to $\bar{y}_n$ and $\hat{\bar{Y}}$, we introduce assumptions $\rho_{xz1} = \rho_{yz1}$, $\rho_{xz2} = \rho_{yz2}$, which are intuitive assumption, considered, for example, by Cochran (1977) and Feng and Zou (1997).

Table 1. Optimum values of μ and percent relative efficiencies of the estimator Δ with respect to $\overline{y}_n$ and $\hat{\hat{\overline{Y}}}$ for f = 0.1.

ρ_{z1z2} ↓	ρ_{yz1} ↓	ρ_{yz2} →	0.5						
		ρ_{yx} →	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.3	0.5	μ^*	0.8015	0.8421	0.9058	*	*	*	*
		E_1	148.61	150.60	152.73	-	-	-	-
		E_2	144.80	143.61	141.36	-	-	-	-
	0.7	μ^*	0.7032	0.7319	0.7746	0.8385	0.9393	*	*
		E_1	200.68	204.61	209.74	215.80	221.31	-	-
		E_2	195.54	195.12	194.13	191.82	186.16	-	-
	0.9	μ^*	0.6099	0.6329	0.6663	0.7143	0.7846	0.8929	*
		E_1	319.16	327.21	338.28	352.94	371.49	392.07	-
		E_2	310.99	312.03	313.10	313.72	312.50	304.95	-
0.5	0.5	μ^*	0.8625	0.9143	*	*	*	*	*
		E_1	131.45	132.63	-	-	-	-	-
		E_2	128.09	126.48	-	-	-	-	-
	0.7	μ^*	0.7500	0.7834	0.8342	0.9135	*	*	*
		E_1	170.64	173.50	177.02	180.54	-	-	-
		E_2	166.27	165.45	163.84	160.48	-	-	-
	0.9	μ^*	0.6578	0.6832	0.7204	0.7747	0.8565	0.9891	*
		E_1	245.03	250.50	257.86	267.20	277.92	285.66	-
		E_2	238.76	238.88	238.67	237.51	233.78	222.18	-
0.7	0.5	μ^*	0.9402	*	*	*	*	*	*
		E_1	117.38	-	-	-	-	-	-
		E_2	114.38	-	-	-	-	-	-
	0.7	μ^*	0.8015	0.8421	0.9058	*	*	*	*
		E_1	148.61	150.60	152.73	-	-	-	-
		E_2	144.80	143.61	141.36	-	-	-	-
	0.9	μ^*	0.7032	0.7319	0.7746	0.8385	0.9393	*	*
		E_1	200.68	204.61	209.74	215.80	221.31	-	-
		E_2	195.54	195.12	194.13	191.82	186.16	-	-
0.9	0.5	μ^*	*	*	*	*	*	*	*
		E_1	-	-	-	-	-	-	-
		E_2	-	-	-	-	-	-	-
	0.7	μ^*	0.8625	0.9143	*	*	*	*	*
		E_1	131.45	132.63	-	-	-	-	-
		E_2	128.09	126.48	-	-	-	-	-
	0.9	μ^*	0.7500	0.7834	0.8342	0.9135	*	*	*
		E_1	170.64	173.50	177.02	180.54	-	-	-
		E_2	166.27	165.45	163.84	160.48	-	-	-

Table 2. Optimum values of μ and percent relative efficiencies of the estimator Δ with respect to $\overline{y}_n$ and $\hat{\hat{Y}}$ for f = 0.1.

ρ_{z1z2} ↓	ρ_{yz1} ↓	ρ_{yz2} →	0.7						
		ρ_{yx} →	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.3	0.5	μ^*	0.7032	0.7319	0.7746	0.8385	0.9393	*	*
		E_1	200.68	204.61	209.74	215.80	221.31	-	-
		E_2	195.54	195.12	194.13	191.82	186.16	-	-
	0.7	μ^*	0.6099	0.6329	0.6663	0.7143	0.7846	0.8929	*
		E_1	319.16	327.21	338.28	352.94	371.49	392.07	-
		E_2	310.99	312.03	313.10	313.72	312.50	304.95	-
	0.9	μ^*	0.4682	0.4879	0.5162	0.5566	0.6150	0.7028	0.8432
		E_1	1134.4	1175.1	1233.1	1314.4	1429.2	1594.2	1831.7
		E_2	1105.3	1120.6	1141.3	1168.4	1202.2	1239.9	1257.7
0.5	0.5	μ^*	0.7500	0.7834	0.8342	0.9135	*	*	*
		E_1	170.64	173.50	177.02	180.54	-	-	-
		E_2	166.27	165.45	163.84	160.48	-	-	-
	0.7	μ^*	0.6578	0.6832	0.7204	0.7747	0.8565	0.9891	*
		E_1	245.03	250.50	257.86	267.20	277.92	285.66	-
		E_2	238.76	238.88	238.67	237.51	233.78	222.18	-
	0.9	μ^*	0.5533	0.5745	0.6052	0.6488	0.7118	0.8066	0.9568
		E_1	475.73	489.52	508.81	535.09	570.38	616.27	663.09
		E_2	463.56	466.81	470.94	475.64	479.80	479.32	455.28
0.7	0.5	μ^*	0.8015	0.8421	0.9058	*	*	*	*
		E_1	148.61	150.60	152.73	-	-	-	-
		E_2	144.80	143.61	141.36	-	-	-	-
	0.7	μ^*	0.7032	0.7319	0.7746	0.8385	0.9393	*	*
		E_1	200.68	204.61	209.74	215.80	221.31	-	-
		E_2	195.54	195.12	194.13	191.82	186.16	-	-
	0.9	μ^*	0.6099	0.6329	0.6663	0.7143	0.7846	0.8929	*
		E_1	319.16	327.21	338.28	352.94	371.49	392.07	-
		E_2	310.99	312.03	313.10	313.72	312.50	304.95	-
0.9	0.5	μ^*	0.8625	0.9143	*	*	*	*	*
		E_1	131.45	132.63	-	-	-	-	-
		E_2	128.09	126.48	-	-	-	-	-
	0.7	μ^*	0.7500	0.7834	0.8342	0.9135	*	*	*
		E_1	170.64	173.50	177.02	180.54	-	-	-
		E_2	166.27	165.45	163.84	160.48	-	-	-
	0.9	μ^*	0.6578	0.6832	0.7204	0.7747	0.8565	0.9891	*
		E_1	245.03	250.50	257.86	267.20	277.92	285.66	-
		E_2	238.76	238.88	238.67	237.51	233.78	222.18	-

Table 3. Optimum values of μ and percent relative efficiencies of the estimator Δ with respect to $\bar{y}_n$ and $\hat{\hat{\bar{Y}}}$ for f = 0.1.

ρ_{z1z2} ↓	ρ_{yz1} ↓	ρ_{yz2} →	0.9						
		ρ_{yx} →	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.3	0.5	μ^*	0.6099	0.6329	0.6663	0.7143	0.7846	0.8929	*
		E_1	319.16	327.21	338.28	352.94	371.49	392.07	-
		E_2	310.99	312.03	313.10	313.72	312.50	304.95	-
	0.7	μ^*	**0.4682**	0.4879	0.5162	0.5566	0.6150	0.7028	0.8432
		E_1	1134.4	1175.1	1233.1	1314.4	1429.2	1594.2	**1831.7**
		E_2	1105.3	1120.6	1141.3	1168.4	1202.2	1239.9	**1257.7**
	0.9	μ^*	*	*	*	*	*	*	*
		E_1	-	-	-	-	-	-	-
		E_2	-	-	-	-	-	-	-
0.5	0.5	μ^*	0.6578	0.6832	0.7204	0.7747	0.8565	0.9891	*
		E_1	245.03	250.50	257.86	267.20	277.92	285.66	-
		E_2	238.76	238.88	238.67	237.51	233.78	222.18	-
	0.7	μ^*	0.5533	0.5745	0.6052	0.6488	0.7118	0.8066	0.9568
		E_1	475.73	489.52	508.81	535.09	570.38	616.27	663.09
		E_2	463.56	466.81	470.94	475.64	479.80	479.32	455.28
	0.9	μ^*	*	*	*	*	*	*	*
		E_1	-	-	-	-	-	-	-
		E_2	-	-	-	-	-	-	-
0.7	0.5	μ^*	0.7032	0.7319	0.7746	0.8385	0.9393	*	*
		E_1	200.68	204.61	209.74	215.80	221.31	-	-
		E_2	195.54	195.12	194.13	191.82	186.16	-	-
	0.7	μ^*	0.6099	0.6329	0.6663	0.7143	0.7846	0.8929	*
		E_1	319.16	327.21	338.28	352.94	371.49	392.07	-
		E_2	310.99	312.03	313.10	313.72	312.50	304.95	-
	0.9	μ^*	0.4682	0.4879	0.5162	0.5566	0.6150	0.7028	0.8432
		E_1	1134.4	1175.1	1233.1	1314.4	1429.2	1594.2	1831.7
		E_2	1105.3	1120.6	1141.3	1168.4	1202.2	1239.9	1257.7
0.9	0.5	μ^*	0.7500	0.7834	0.8342	0.9135	*	*	*
		E_1	170.64	173.50	177.02	180.54	-	-	-
		E_2	166.27	165.45	163.84	160.48	-	-	-
	0.7	μ^*	0.6578	0.6832	0.7204	0.7747	0.8565	0.9891	*
		E_1	245.03	250.50	257.86	267.20	277.92	285.66	-
		E_2	238.76	238.88	238.67	237.51	233.78	222.18	-
	0.9	μ^*	0.5533	0.5745	0.6052	0.6488	0.7118	0.8066	0.9568
		E_1	475.73	489.52	508.81	535.09	570.38	616.27	663.09
		E_2	463.56	466.81	470.94	475.64	479.80	479.32	455.28

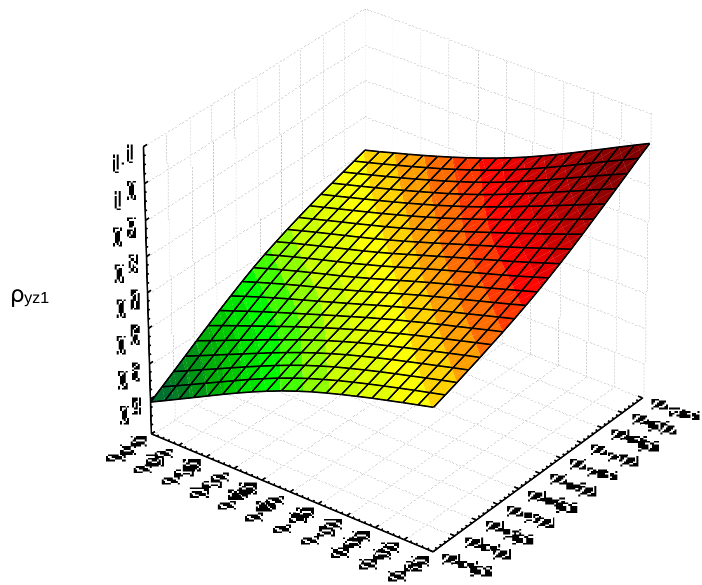

Figure 1. 3D Surface Plot of μ against ρ_{yz1} and ρ_{yz2} for ρ_{yx} = 0.3 and ρ_{z1z2} = 0.7

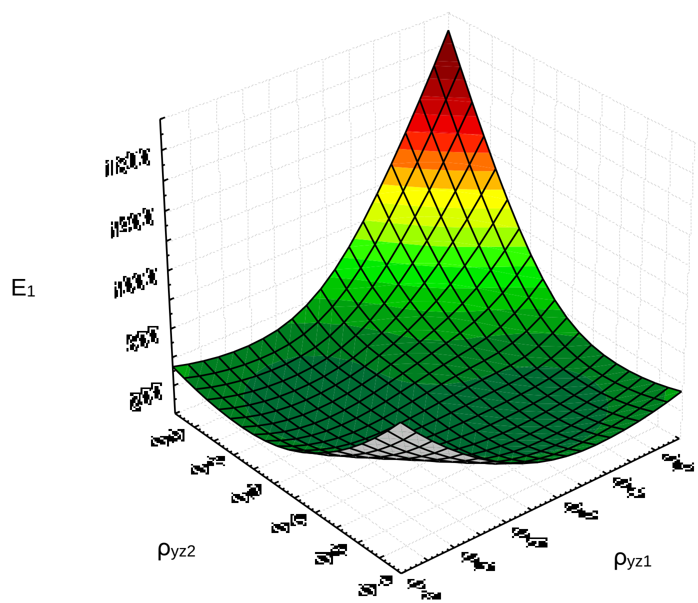

Figure 2. 3D Surface Plot of E_1 against ρ_{yz1} and ρ_{yz2} for ρ_{yx} = 0.5 and ρ_{z1z2} = 0.7

Remark 3:"*" in the Tables 1-3 indicates that μ^* do not exist for the corresponding combinations of correlations.

The following results may be extracted from Tables 1-3:

(a) For the fixed value of ρ_{yx} and ρ_{z1z2}, the value of μ decreases but E_1 and E_2 increases with the increasing values of ρ_{yz1} and ρ_{yz2}. This phenomenon is expected as availability of highly correlated auxiliary variables results in reducing the cost of survey and enhancing the precision of estimates.

(b) For the fixed value of ρ_{z1z2}, ρ_{yz1} and ρ_{yz2}, the value μ, E_1 and E_2 increases with the increasing values of ρ_{yx}. This behaviour indicates that when the study characters over the occasions are highly correlated, a larger fresh sample at the current occasion is required but the efficiency of estimates increases.

(c) For fixed values of ρ_{yz1}, ρ_{yz2} and ρ_{yx}, the values of μ are increasing whereas decreasing trends are observed in the efficiencies E_1 and E_2.

(d) The lowest value of μ is observed as 0.4682, which indicates that the fraction of fresh sample to be drawn at the current occasion is as low as 46% of the total sample size.

(e) Percent relative gain in efficiencies E_1 and E_2 seems to be appreciably high under optimal conditions. Highest values of E_1 and E_2 are observed as 1831.7 and 1257.7 respectively.

(f) A close perusal of Figure 1 suggests that for the fixed value of ρ_{yx} and ρ_{z1z2}, the value of μ decreases with the increasing values of ρ_{yz1} and ρ_{yz2}. This behaviour supports the fact that if highly correlated auxiliary variables are available, the less fresh sample is required to be drawn at the current occasion, which subsequently reduces the cost of survey.

(h) It can be clearly seen from Figure 2 that for the fixed value of ρ_{yx} and ρ_{z1z2}, E_1 increases with the increasing values of ρ_{yz1} and ρ_{yz2}. This phenomenon justifies the use of two auxiliary variables, which results in increasing the precision of estimates.

6. Illustrations using real life data

To illustrate the comparison of relative efficiency of the proposed estimator with respect to $\bar{y}_n$ and $\hat{\bar{Y}}$, using real life approach, the data from the Census of India (2001) and (2011) were taken into account. We define the variables as

x (y): the total number of workers in villages in the district of Ranchi, India in 2001 (2011)

z_1: the total number of literate people in villages in the district of Ranchi, India.

z_2: the total number of females in the district of Ranchi, India.

Using successive sampling strategy defined in Section 2 for the above data set we have taken n (sample drawn at first occasion) = 70, m (matched portion of the sample over two occasions) = 35 and u (fresh sample drawn at second occasion) = 35. The values of the different estimators computed from the sample along with their corresponding mean square errors and efficiency of the proposed estimator Δ with respect to $\bar{y}_n$ and $\hat{\hat{Y}}$ have been shown in the Table 4.

Table 4. Relative Efficiency (%) of estimator Δ with respect to $\bar{y}_n$ and $\hat{\hat{Y}}$ using real life data

Estimators	Estimates	MSE	% Efficiency
Δ	481	341.86	100
$\bar{y}_n$	465	1926.13	563.42
$\hat{\hat{Y}}$	422	1360.48	397.96

The above table validates the conclusions observed from the Tables 1, 2 and 3 that the proposed estimator Δ is more efficient than $\bar{y}_n$ and $\hat{\hat{Y}}$ with maximum gain in efficiency occurring while comparing it with mean per unit estimator, which is the expected phenomenon.

7. Conclusion

In the context of the preceding interpretations, it may be concluded that the use of two auxiliary variables for the estimation of population mean at the current occasion in two-occasion successive sampling is highly appreciable, as demonstrated through empirical results. It is also observed and justified through this study that the use of exponential type estimator is highly rewarding in terms of precision. Hence, the proposed estimator Δ may be recommended for its practical use by survey practitioners.

Acknowledgement

Authors are thankful to the honourable referees and University Grants Commission, New Delhi (PDFWM-2014-15-GE-ASS-28217) for providing necessary infrastructure to carry out the present work.

REFERENCES

BEEVI, N. T., CHANDRAN, C., (2017). Efficient family of ratio-type estimators for mean estimation in successive sampling using auxiliary information on both occasions. Statistics In Transition – New Series, 18 (2), pp. 227–246.

BIRADAR, R. S., SINGH, H. P., (2001). Successive sampling using auxiliary information on both occasions. Cal. Statist. Assoc. Bull. 51, pp. 243–251.

CENSUS OF INDIA, (2001). www.cesusindiagov.in.

CENSUS OF INDIA, (2011). www.cesusindiagov.in.

CHATURVEDI, D. K., TRIPATHI, T. P., (1983). Estimation of population ratio on two occasions using multivariate auxiliary information. Journal of Indian Statistical Association 21, pp. 113–120.

COCHRAN, W. G., (1977). Sampling Techniques, Wiley Eastern Limited, New Delhi, III Edition.

DAS, A. K., (1982). Estimation of population ratio on two occasions, Jour Ind. Soc. Agr. Statist. 34, pp. 1–9.

FENG, S., ZOU, G., (1997). Sample rotation method with auxiliary variable. Communications in Statistics-Theory and Methods, 26 (6), pp. 1497–1509.

GUPTA, P. C., (1979). Sampling on two successive occasions. Jour. Statist. Res. 13, pp. 7–16.

JESSEN, R. J., (1942). Statistical Investigation of a Sample Survey for obtaining farm facts, Iowa Agricultural Experiment Station Research Bulletin No. 304, Ames, Iowa, USA, pp. 1–104.

KARNA, J. P., NATH, D. C., (2016). Rotation sampling scheme using transformed auxiliary variable. Journal of Statistics & Management Systems, 19 (6), pp. 739–754.

PATTERSON, H. D., (1950). Sampling on successive occasions with partial replacement of units, Journal of the Royal Statistical Society, 12, pp. 241–255.

RALTE, Z., DAS, G., (2015). Ratio-to-regression estimator in successive sampling using one auxiliary variable. Statistics in Transition – new series, 16 (2), pp. 183–202.

RAO, J. N. K., GRAHAM, J. E., (1964). Rotation designs for sampling on repeated occasions, Journal of the American Statistical Association, 59, pp. 492–509.

REDDY, V. N., (1978). A study on the use of prior knowledge on certain population parameters. Sankhya, 40, C, pp. 29–37.

SEN, A. R., (1971). Successive sampling with two auxiliary variables, Sankhya, 33, Series B, pp. 371–378.

SEN, A. R., (1972). Successive sampling with p $(p \geq 1)$ auxiliary variables, Ann. Math. Statist., 43, pp. 2031–2034.

SEN, A. R., (1973). Some theory of sampling on successive occasions, Australian Journal of Statistics, 15, pp. 105–110.

SINGH, G. N., (2005). On the use of chain-type ratio estimator in successive sampling, Statistics in Transition, 7, pp. 21–26.

SINGH, G. N., KARNA, J. P., (2009a). Estimation of Population mean on the current in two-occasion successive sampling. Metron, 67(1), 2009, pp. 87–103.

SINGH, G. N., KARNA, J. P., (2009b). Search of effective rotation patterns in presence of auxiliary information in successive sampling over two occasions. Statistics in Transition-new series, 10, pp. 209, 59–73.

SINGH, H. P., PAL, S. K., (2015). An efficient effective rotation pattern in successive sampling over two occasions. Communication in Statistics: Theory and Methods, 45 (17), pp. 5017–5027.

SINGH, H. P., PAL, S. K., (2017). A class of exponential type estimators of a general parameter. Communication in Statistics – Theory and Methods, 46 (8), pp. 3957–3984.

SUKHATME, P. V., SUKHATME, B. V., SUKHATME, S., ASOK, C., (1984). Sampling theory of surveys with applications. Iowa State University Press, Ames, Iowa (USA) and Indian Society of Agricultural Statistics, New Delhi (India),III Revised Edition.

UPADHYAYA, L. N., SINGH, H. P., CHATTERJEE, S., YADAV, R., (2011). Improved ratio and product exponential type estimators. J. Stat. Theo. Pract. 5 (2), pp. 285–302.

YADAV, S. K., KADILAR, C., (2013). Improved exponential type ratio estimator of population variance. Revis. Colum. de Estadist. 36 (1), pp. 145–152.

APPENDIX

Bias of estimator Δ_u

$$B(\Delta_u) = E\left[\Delta_u - \bar{Y}\right]$$

$$= E\left[\bar{y}_u \exp\left(\frac{\bar{Z}_1 - \bar{z}_{1u}}{\bar{Z}_1 + \bar{z}_{1u}}\right) \exp\left(\frac{\bar{Z}_2 - \bar{z}_{2u}}{\bar{Z}_2 + \bar{z}_{2u}}\right) - \bar{Y}\right]$$

$$= E\left[(1+e_1)\bar{Y}\exp\left(\frac{-e_5}{2+e_5}\right)\exp\left(\frac{-e_7}{2+e_7}\right) - \bar{Y}\right], \text{ from eq. (4)}$$

$$= \bar{Y}E\left[e_1\exp\left(\frac{-e_5}{2+e_5}\right)\exp\left(\frac{-e_7}{2+e_7}\right)\right]$$

$$= \bar{Y}E\left[e_1\exp\left\{-\frac{e_5}{2}\left(1+\frac{e_5}{2}\right)^{-1}\right\}\exp\left\{-\frac{e_7}{2}\left(1+\frac{e_7}{2}\right)^{-1}\right\}\right]$$

$$= \bar{Y}E\left[e_1\left\{1-\frac{e_5}{2}\left(1+\frac{e_5}{2}\right)^{-1}+\frac{1}{2}.\frac{e_5^2}{4}\left(1+\frac{e_5}{2}\right)^{-2}+...\right\}\left\{1-\frac{e_7}{2}\left(1+\frac{e_7}{2}\right)^{-1}+\frac{1}{2}.\frac{e_7^2}{4}\left(1+\frac{e_7}{2}\right)^{-2}+...\right\}\right]$$

Retaining the terms up to the first order of approximations, we get

$$B(\Delta_u) = \bar{Y}E\left[e_1\left\{1-\frac{e_5}{2}+\frac{3e_5^2}{8}\right\}\left\{1-\frac{e_7}{2}+\frac{3e_7^2}{8}\right\}\right]$$

$$= \bar{Y}E\left[e_1-\frac{e_5}{2}-\frac{e_7}{2}+\frac{3e_5^2}{8}+\frac{3e_7^2}{8}-\frac{e_1e_7}{2}-\frac{e_1e_5}{2}+\frac{e_5e_7}{4}\right]$$

$$= \bar{Y}E\left[e_1-\frac{e_5}{2}-\frac{e_7}{2}+\frac{3e_5^2}{8}+\frac{3e_7^2}{8}-\frac{e_1e_7}{2}-\frac{e_1e_5}{2}+\frac{e_5e_7}{4}\right]$$

Taking expectations as discussed in Section 3.1 and using Note 1, we have

$$B(\Delta_u) = \frac{1}{\bar{Y}}\left(\frac{1}{u}-\frac{1}{N}\right)\left(\frac{3}{4}-\frac{\rho_{yz2}}{2}-\frac{\rho_{yz1}}{2}+\frac{\rho_{z1z2}}{4}\right)S_y^2$$

Bias of estimator Δ_m

$$B(\Delta_m) = E\left[\Delta_m - \bar{Y}\right]$$

$$= E\left[\left[(1+e_2)\bar{Y} + (1+e_9)(e_4 - e_3)(1+e_{10})^{-1}\beta_{yx}\bar{X}\right]\exp\left(\frac{-e_6}{2+e_6}\right)\exp\left(\frac{-e_8}{2+e_8}\right) - \bar{Y}\right], \text{ from eq (5)}$$

Expanding the exponentials as in the case of bias of estimator Δ_u and retaining the terms up to first order of approximations, we get

$$B(\Delta_m) = E\left[\left\{(1+e_2)\bar{Y} + (e_4 - e_3 + e_4e_9 - e_3e_9 - e_4e_{10} + e_3e_{10})\beta_{yx}\bar{X}\right\}\left\{1 - \frac{e_8}{2} + \frac{3}{8}e_8^2 - \frac{e_6}{2} + \frac{e_6e_8}{4} + \frac{3}{8}e_6^2\right\} - \bar{Y}\right]$$

$$= E\left[\left(\frac{3}{8}e_8^2 + \frac{3}{8}e_6^2 - \frac{e_2e_8}{2} - \frac{e_2e_6}{2} + \frac{e_6e_8}{4}\right)\bar{Y} + \left(e_4e_9 - e_3e_9 - e_4e_{10} + e_3e_{10} - e_4e_8 - \frac{e_4e_6}{2} + \frac{e_3e_8}{2} + \frac{e_3e_6}{2}\right)\beta_{yx}\bar{X}\right]$$

Now, taking expectations as discussed in Section 3.1 and using Note 1, we have

$$B(\Delta_m) = \frac{1}{\bar{Y}}\left(\frac{1}{n} - \frac{1}{N}\right)\left(\frac{3}{4} - \frac{\rho_{yz2}}{2} - \frac{\rho_{yz1}}{2} + \frac{\rho_{z1z2}}{4}\right) + \left(\frac{1}{m} - \frac{1}{n}\right)\left(\frac{\alpha_{2100}}{S_x^2} + \frac{\alpha_{3000}S_{yx}}{S_x^4}\right)$$

$$- \frac{1}{2}\left(\frac{1}{n} - \frac{1}{N}\right)\frac{\rho_{xz2}S_{yx}}{\bar{X}}$$

- The mean square errors of the estimators Δ_u and Δ_m have been obtained in the similar manner as the bias of the estimators.

- The variance of the estimator natural successive sampling estimator $\hat{\bar{Y}}$ has been discussed in detail in Sukhatme et al. (1984), pages 256-260.