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LINDLEY PARETO DISTRIBUTION

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ABSTRACT

In this paper, we introduce a new Lindley Pareto distribution, which offers a more flexible model for modelling lifetime data. Some of its mathematical properties like density function, cumulative distribution, mode, mean, variance, and Shannon entropy are established. A simulation study is carried out to examine the bias and mean square error of the maximum likelihood estimators of the unknown parameters. Three real data sets are fitted to illustrate the importance and the flexibility of the proposed distribution.

Key words: T-X family, Lindley distribution, Pareto distribution.

1 Introduction

Statistical distributions (Lifetime distributions) are commonly applied to describe real world phenomena and are most frequently used in many applied sciences such as reliability, engineering, actuarial sciences, demography, economics, hydrology, biological studies, insurance, medicine and finance. Recently this issue has received much attention from researchers and practitioners. The quality and effectiveness of the procedures used in a statistical analysis are determined by the assumed probability distribution. Recently, one parameter Lindley distribution has attracted the researchers for its use in stress-strength reliability modelling, and it has been observed in several papers that this distribution has performed excellently. The Lindley distribution was introduced by Lindley (1958) as a new distribution useful to analyze lifetime data. Sankaran (1970) introduced the discrete Poisson-Lindley distribution by combining the Poisson and Lindley distributions. Many generalizations of the Lindley distribution have been proposed in recent years. Asgharzadeh et al. (2013), Ghitany et al. $(2008a, 2008b)$ rediscovered and

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studied the new generalizations of Lindley distribution, what they derived is known as Zero-truncated Poisson- Lindley and Pareto Poisson-Lindley distributions. There still remain many important problems where the real data does not follow any of the existing probability distributions. Considerable effort has been expended in the development of large classes of new probability distributions along with relevant statistical methodologies.

Furthermore, Pareto distribution was pioneered by V. Pareto (1896) to explore the unequal distribution of wealth. It is widely used in actuarial science. (e.g. reinsurance) because of its heavy tail properties. To add flexibility to the Pareto distribution, various generalizations of the distribution have been derived, including the generalized Pareto distribution (Pickands, 1975), the beta-Pareto distribution (Akinsete et al., 2008), and the beta generalized Pareto distribution (Mahmoudi, 2011).

The mixing method is one of the most important ideas for obtaining a new distribution. For example, Sharma and Shanker (2013) used a mixture of exponential (θ) and gamma $(2,\theta)$ to create a two-parameter Lindley distribution. Another example includes Zakerzadeh and Dolati (2010), who used gamma (α,θ) and gamma $(\alpha+1,\theta)$ to create a generalized Lindley distribution. Recently, Zeghdoudi and Nedjar $(2016a, 2016b)$ introduced a new distribution, named gamma Lindley distribution, based on mixtures of gamma $(2,\theta)$ and one-parameter Lindley distributions.

Gomes-Silva et al. (2017) introduce a new generator of continuous distributions with one extra positive parameter called the odd Lindley-G family. Some special cases are given (Odd Lindley Weibull, Odd Lindley Kumaraswamy, Odd Lindley half-logistic and Odd Lindley Burr XII), where the hazard rate function of the Odd Lindley Burr XII distribution can be constant, increasing, decreasing, unimodal or bathtub shape. For more details on this last distribution function we refer the reader to Abouelmagd et al. (2018).

In addition, the cumulative distribution function (cdf) of the T-X family of distributions defined by Alzaatreh et al. (2013) is given by

$$G(x) = \int_0^{W(F(x))} r(t)\,dt, \tag{1}$$

where $W(F(x))$ satisfies the following conditions:

- $W(F(x)) \in [a,b]$,

-$W(F(x))$ is differentiable and monotonically non-decreasing,

- $W(F(x)) \to a$ as $x \to -\infty$ and $W(F(x)) \to b$ as $x \to \infty$.

In this paper, we propose a new wider class of continuous distributions called the Lindley Pareto (LP for short) by taking $W(F(x)) = \frac{F(x)}{1-F(x)}$ and $r(t) = \frac{\theta^2}{1+\theta}(1+t)\exp(-\theta t)$, $x>0$, $\theta>0$, where $F(x)$ corresponding to Pareto distribution: $F(x) = 1-\left(\frac{\alpha}{x}\right)^k$, $x>\alpha$. Its cdf is given by

$$G(x) = 1 - \frac{\left(\alpha^k + x^k\theta\right)}{(\theta+1)\,\alpha^k}\exp\left(-\theta\left(\frac{x^k}{\alpha^k}-1\right)\right), \tag{2}$$

with corresponding density

$$g(x) = \frac{k\theta^2 e^{\theta} x^{2k-1}}{(\theta+1)\,\alpha^{2k}}\exp\left(-\theta\left(\frac{x}{\alpha}\right)^k\right), x>\alpha. \tag{3}$$

We can see the plots of the density function and the distribution function of LP distribution for some parameter values in Appendix 1. We refer to the cdf in equation (1) as Lindley Pareto (LP) distribution with parameters θ, α, k, which we denote by $LP(\theta,\alpha,k)$. The objective of this work is to study some mathematical properties of the Lindley Pareto model with the hope that it will attract wider applications in reliability, engineering and other areas of research.

The LP distribution is motivated by the following: the LP distribution use may be restricted to the tail of a distribution, but it is easy to apply. The formulas of the mean, variance, mean deviation, entropy and the quantile function are simple in form and may be used as quick approximations in many cases. Also, the LP distribution can be viewed as a special case of odd Lindley-G family introduced by Gomes-Silva et al.(2017). Also, this new distribution has advantages including a number of parameters (three) which we can modelled physical phenomena inspired in Cooray (2006). Furthermore, LP distribution can be used quite effectively in analyzing many real lifetime data sets: application to waiting times in a queue, Wheaton River Data and application to bladder cancer patients. Moreover, the actuarial literature has discussed hundreds of univariate continuous distributions, of which log-normal, Weibull, multi-parameter Pareto, gamma distributions as well as others.

The remainder of the article is unfolded as follows: in Section 2, various properties of LP distribution are examined, including survival and hazard functions, reliability, mean deviation, entropy and quantile function. The model parameters are estimated via the maximum likelihood estimates (MLEs) and some simulations are proposed in Section 3. In Section 4, the importance and potentiality of LP distribution are shown using three real lifetime data sets. Finally, some concluding notes are provided in Section 5.

2 Main properties

2.1 Survival and hazard functions

The survival and hazard functions corresponding to the cdf defined in (1) are given by

$$S(x) = 1 - G(x) = \frac{\left(\alpha^k + x^k\theta\right)}{(\theta+1)\,\alpha^k}\exp\left(-\theta\left(\frac{x^k}{\alpha^k}-1\right)\right)$$

and

$$h(x) = \frac{k\theta^2 x^{2k-1}}{\alpha^k(\theta x^k + \alpha^k)}.$$

2.2 Reliability

The measure of reliability has many applications, especially in the area of engineering. The component fails at the instant that the random stress X_2 applied to it exceeds the random strength X_1, and the component will function satisfactorily whenever $X_1 > X_2$. Hence, $R = P[X_2 < X_1]$ is a measure of component reliability. We derive the reliability R when X_1 and X_2 have independent $LP(\theta_1,\alpha,k)$ and $LP(\theta_2,\alpha,k)$ distributions. The reliability is defined by

$$R = \int_0^\infty g_1(x)\,G_2(x)\,dx = \sum_{i,j,k,l=0} \frac{p_{i,j}(\theta_1)\,q_{k,l}(\theta_2)}{i+j+k+l+2},$$

where

$$p_{i,j}(\theta_1) = \frac{(-1)^j\,\theta_1^{2+j}\Gamma(i+j+3)}{i!j!\,(\theta_1+1)\,\Gamma(j+3)}$$

and

$$q_{k,l}(\theta_2) = \frac{(-1)^l\,\theta_2^{2+l}\Gamma(k+l+3)}{k!l!\,(\theta_2+1)\,(k+l+1)\,\Gamma(l+3)}.$$

2.3 Mean deviations

The deviation from the mean and the median are used to measure the dispersion and spread in a population from the centre. If the median is denoted by M, then the mean deviation from the mean, $D(\mu)$, and the mean deviation from the median, $D(M)$, can be written as

$$D(\mu) = \int_{\alpha}^{\infty} |x-\mu|\, g(x)\, dx = 2\mu G(\mu) - 2\int_{\alpha}^{\mu} x g(x)\, dx,$$

$$D(M) = \int_{\alpha}^{\infty} |x-M|\, g(x)\, dx = \mu - 2\int_{\alpha}^{M} x g(x)\, dx.$$

Consider the integral

$$\int_{\alpha}^{b} x g(x)\, dx = \int_{\alpha}^{b} \frac{k\theta^2 e^{\theta}}{(\theta+1)\,\alpha^{2k}} x^{2k} \exp\left(-\theta\left(\frac{x}{\alpha}\right)^k\right) dx \; = \left(-\frac{e^{\theta}}{(\theta+1)} \frac{\alpha\Gamma(\frac{2k+1}{k}, \theta\frac{x^k}{\alpha^k})}{\theta^{\frac{1}{k}}}\right|_{a}^{b}$$

we obtain,

$$\begin{aligned} D(\mu) &= 2\mu G(\mu) - \int_{\alpha}^{\mu} x g(x)\, dx \\ &= 2\mu G(\mu) - \frac{\alpha e^{\theta}}{(\theta+1)\,\theta^{\frac{1}{k}}} \left(\Gamma(\frac{1}{k}+2, \theta) - \Gamma(\frac{1}{k}+2, \theta\left(\frac{\mu}{\alpha}\right)^k)\right), \end{aligned}$$

$$\begin{aligned} D(M) &= \mu - 2\int_{\alpha}^{M} x g(x)\, dx \\ &= \mu - 2\frac{\alpha e^{\theta}}{(\theta+1)\,\theta^{\frac{1}{k}}} \left(\Gamma(\frac{1}{k}+2, \theta) - \Gamma(\frac{1}{k}+2, \theta\left(\frac{M}{\alpha}\right)^k)\right). \end{aligned}$$

2.4 Entropy

The entropy of a random variable X is a measure of variation of uncertainty (see, Rényi, 1961), that of the LP distribution is given by

$$I_R(s) = \frac{1}{1-s} \ln\left(\frac{k^s \theta^{2s} e^{s\theta}}{\theta^{\frac{1-s}{k}} s^{\frac{2ks-s+1}{k}} (\theta+1)^s \alpha^{s-1}} \frac{\Gamma\left(\frac{2ks-s+1}{k}, \theta s\right)}{k}\right) \; s>0, s\neq 1.$$

Shannon entropy (Shannon, 1948) for a random variable X with density $g(x)$ is defined as $E\{-\ln(g(x))\}$.

$$E\{-\ln(g(X))\} = \ln k + 2\ln\theta + \theta - \ln(\theta+1) - 2k\ln\alpha + 2kE(\ln x) - \frac{\theta}{\alpha^k} E\left(x^k\right),$$

$$E\left\{-\ln\left(g\left(X\right)\right)\right\}=\theta+\ln k+2\ln\theta-\ln\left(\theta+1\right)+\frac{2\left(1+Ei(\theta)\,e^{-\theta}\right)-\left(\theta^{2}+2\theta+2\right)}{\left(\theta+1\right)}.$$

where, *Ei* is the exponential integral function.

2.5 Quantile function

The quantile function of the LP distribution X is

$$x_{\gamma}=\alpha\left(-\frac{1}{\theta}-\frac{1}{\theta}\mathrm{LAMBERTW(X)}\left(-1,\left(\gamma-1\right)\left(\theta+1\right)e^{-\theta-1}\right)\right)^{\frac{1}{k}},\ 0<\gamma<1, \tag{4}$$

where $\theta,\alpha,k>0$ and $\mathrm{LAMBERTW(X)}$ denotes the negative branch of the $\mathrm{LAMBERTW(X)}$ function ($W(z)\exp\left(W(z)\right)=z$, where z is a complex number). For more details we refer the reader to Lazri and Zeghdoudi (2016).

3 Estimation and Simulation

3.1 Maximum Likelihood Estimates (ML)

Let $X_i\sim LP(\theta,\alpha,k)$, $i=1,...,n$ be n random variables. The ln-likelihood function, $\ln l(x_i;\theta,\alpha,k)$ is:

$$L\left(\Theta\right)=\ln l(x;\theta,\alpha,k)=n\ln k+2n\ln\theta+n\theta-2kn\ln\alpha-n\ln\left(\theta+1\right)+\left(2k-1\right)\sum_{i=1}^{n}\ln x_i-\theta\sum_{i=1}^{n}\left(\frac{x^{k}}{\alpha^{k}}\right).$$

To simplify, we assume that α is known, the derivatives of $L\left(\Theta\right)$ with respect to θ and k are:

$$\frac{dL\left(\Theta\right)}{d\theta}=\frac{2n}{\theta}+n-\frac{n}{\left(\theta+1\right)}-\frac{1}{\alpha^{k}}\sum_{i=1}^{n}x_i^{k}, \tag{5}$$

$$\frac{dL\left(\Theta\right)}{dk}=\frac{n}{k}-2n\ln\alpha\ +2\sum_{i=1}^{n}\ln x_i-\frac{\theta}{\alpha^{k}}\sum_{i=1}^{n}x_i^{k}\ln x_i+\frac{\theta\ln\alpha}{\alpha^{k}}\sum_{i=1}^{n}x_i^{k}. \tag{6}$$

The two equations (5) and (6) cannot be solved directly, we must used the Fisher scoring method. We have

$$\left[\begin{array}{cc}\frac{\partial^{2}L(\Theta)}{\partial\theta^{2}} & \frac{\partial^{2}L(\Theta)}{\partial\theta\partial k}\\ \frac{\partial^{2}L(\Theta)}{\partial k\partial\theta} & \frac{\partial^{2}L(\Theta)}{\partial k^{2}}\end{array}\right]_{\substack{\hat{\theta}=\theta_0\\ \hat{k}=k_0}}\left[\begin{array}{c}\hat{\theta}-\theta_0\\ \hat{k}-k_0\end{array}\right]=\left[\begin{array}{c}\frac{dL(\Theta)}{d\theta}\\ \frac{dL(\Theta)}{dk}\end{array}\right]_{\substack{\hat{\theta}=\theta_0\\ \hat{k}=k_0}}, \tag{7}$$

where,

$$\frac{\partial^2 L(\Theta)}{\partial \theta^2} = -\frac{2n}{\theta^2} + \frac{n}{(\theta+1)^2},$$

$$\frac{\partial^2 L(\Theta)}{\partial k^2} = \frac{-n}{k^2} - \theta \sum_{i=1}^{n} \left(\frac{x_i^k}{\alpha^k}\right) \ln^2 \frac{x_i}{\alpha},$$

and

$$\frac{\partial^2 L(\Theta)}{\partial \theta \partial k} = \frac{\partial^2 L(\Theta)}{\partial k \partial \theta} = -\sum_{i=1}^{n} \left(\ln \frac{x_i}{\alpha}\right) \left(\frac{x_i^k}{\alpha^k}\right).$$

The equation (7) can be solved iteratively where θ_0, k_0 are the initial values of θ, k.

Existence and uniqueness of the MLE's

Lemma 1. For any given $\eta > 0$, there exists a compact subset $K \equiv K(\eta) \subset (0,\infty) \times (0,\infty)$ such that

$$\{(\theta,k) : L(\Theta) \geq -\eta\} \subset K. \tag{*}$$

Theorem 2. Suppose that $X_i \sim LP(\theta,\alpha,k)$, $i = 1,...,n$, then the MLEs of parameters θ and k of Pareto Lindley distribution uniquely exist.

Proof. We need only to show that the MLEs of parameters θ and k uniquely exist. According to the results of Mäkeläinen et al. (1981), in order to show the existence and uniqueness of the MLEs of θ and k , it is sufficient to verify the following two conditions:
i) For any given $\eta > 0$, $(*)$ holds.
ii) The Hessian matrix of $L(\Theta)$ is negative definite at every point $(\theta,k) \in (0,\infty) \times (0,\infty)$. Condition i is certainly satisfied by Lemma 1. Therefore, to prove the theorem, we need only to show ii. Then,

$$x^t H x = -2x_1 x_2 \sum_{i=1}^{n} \left(\ln \frac{x_i}{\alpha}\right) \left(\frac{x_i^k}{\alpha^k}\right) + \left(-\frac{2n}{\theta^2} + \frac{n}{(\theta+1)^2}\right) x_1^2 + \left(\frac{-n}{k^2} - \theta \sum_{i=1}^{n} \left(\frac{x_i^k}{\alpha^k}\right) \ln^2 \frac{x_i}{\alpha}\right) x_2^2,$$

where $x^t = (x_1 \;\; x_2)$ and $H = \begin{bmatrix} \frac{\partial^2 L(\Theta)}{\partial \theta^2} & \frac{\partial^2 L(\Theta)}{\partial \theta \partial k} \\ \frac{\partial^2 L(\Theta)}{\partial k \partial \theta} & \frac{\partial^2 L(\Theta)}{\partial k^2} \end{bmatrix}$, we can check that $x^t H x \leq 0$, (H is negative definite).

3.2 Simulation

In this section, we investigate the behaviour of the ML estimators for a finite sample size (n). A simulation study consisting of the following steps is being carried out for each quadruplets (θ,α,k,n), where $\theta = 0.5, 1, 2$, $\alpha = 0.3, 0.5, 1$, $k = 0.75, 1, 2$ and $n = 30, 50, 100$.

- Choose the initial values of θ_0, α_0, k_0 for the corresponding elements of the parameter vector $\Theta = (\theta, \alpha, k)$ to specify LP(θ,α,k) distribution;
- choose sample size n;
- generate N independent samples of size n from LP(θ,α,k);
- compute the ML estimate $\hat{\Theta}_n$ of Θ_0 for each of the N samples;
- compute the mean of the obtained estimators over all N samples,

$$average\ bias\,(\theta) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\Theta}_i - \Theta_0\right),$$

and the average square error

$$MSE\,(\theta) = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\Theta}_i - \Theta_0\right)^2,$$ see Tables 1and 2.

Table 1. *Average bias* of the simulated estimates

	$\theta=0.75\quad\alpha=0.3\quad k=1.5$			$\theta=1.25\quad\alpha=0.3\quad k=2$		
	$bias(\theta)$	$bias(\alpha)$	$bias(k)$	$bias(\theta)$	$bias(\alpha)$	$bias(k)$
n=30	0.2034	0.0192	−0.0768	0.3261	0.0071	0.0210
n=50	0.0788	0.0087	0.02108	0.1460	0.0040	0.0589
n=100	0.0653	0.0058	−0.0066	0.0894	0.0022	−0.0117
	$\theta=1\quad\alpha=1.25\quad k=1.5$			$\theta=1\quad\alpha=2\quad k=5$		
	$bias(\theta)$	$bias(\alpha)$	$bias(k)$	$bias(\theta)$	$bias(\alpha)$	$bias(k)$
n=30	0.5084	0.0494	−0.1130	0.2532	0.0238	−0.0237
n=50	0.1784	0.0269	−0.0220	0.1130	0.0165	0.0132
n=100	0.1048	0.0167	−0.0326	0.0993	0.0066	−0.0567
	$\theta=1.5\quad\alpha=1\quad k=1.25$			$\theta=2\quad\alpha=3\quad k=1.25$		
	$bias(\theta)$	$bias(\alpha)$	$bias(k)$	$bias(\theta)$	$bias(\alpha)$	$bias(k)$
n=30	0.5046	0.0239	0.1046	0.2366	$2.05867\ 10^{-3}$	$1.6487\ 10^{-2}$
n=50	0.3976	0.0117	0.0826	0.0323	$1.5698\ 10^{-3}$	$6.2404\ 10^{-3}$
n=100	0.2004	0.0073	0.0095	0.0789	$3.7259\ 10^{-5}$	$1.9747\ 10^{-3}$
	$\theta=4\quad\alpha=3\quad k=3$			$\theta=1.5\quad\alpha=5\quad k=7$		
	$bias(\theta)$	$bias(\alpha)$	$bias(k)$	$bias(\theta)$	$bias(\alpha)$	$bias(k)$
n=30	1.4481	0.0094	0.5102	0.4280	0.0251	−0.3361
n=50	0.7441	0.0071	0.5010	0.2127	0.0136	−0.0616
n=100	0.6058	0.0033	0.1447	0.0499	0.0069	0.2307
	$\theta=0.5\quad\alpha=0.3\quad k=0.9$			$\theta=1\quad\alpha=0.8\quad k=0.5$		
	$bias(\theta)$	$bias(\alpha)$	$bias(k)$	$bias(\theta)$	$bias(\alpha)$	$bias(k)$
n=30	0.3240	0.0668	−0.0991	0.1971	0.0896	0.0151
n=50	0.1088	0.0394	0.00431	0.1445	0.0680	0.0057
n=100	0.0520	0.0186	−0.0046	0.0521	0.0376	0.0091
	$\theta=0.75\quad\alpha=0.5\quad k=1.25$			$\theta=3\quad\alpha=1.5\quad k=2$		
	$bias(\theta)$	$bias(\alpha)$	$bias(k)$	$bias(\theta)$	$bias(\alpha)$	$bias(k)$
n=30	0.155	0.0355	−0.0150	1.5423	0.0097	0.1401
n=50	0.1595	0.0258	−0.0374	0.9111	0.0066	0.0729
n=100	0.0827	0.0128	−0.0172	0.5035	0.0032	0.0244

Table 2. *Average MSE* of the simulated estimates

	$\theta=0.75\quad\alpha=0.3\quad k=1.5$			$\theta=1.25\quad\alpha=0.3\quad k=2$		
	$MSE(\theta)$	$MSE(\alpha)$	$MSE(k)$	$MSE(\theta)$	$MSE(\alpha)$	$MSE(k)$
n=30	0.0414	$3.6786\ 10^{-4}$	$5.9026\ 10^{-3}$	0.1063	$5.0350\ 10^{-5}$	$4.4275\ 10^{-4}$
n=50	$6.2155\ 10^{-3}$	$7.4996\ 10^{-5}$	$4.4426\ 10^{-4}$	$2.1316\ 10^{-2}$	$1.5794\ 10^{-5}$	$3.3969\ 10^{-3}$
n=100	$4.2580\ 10^{-3}$	$3.3405\ 10^{-5}$	$4.3903\ 10^{-5}$	$7.9979\ 10^{-5}$	$5.0401\ 10^{-6}$	$1.3671\ 10^{-4}$
	$\theta=1\quad\alpha=1.25\quad k=1.5$			$\theta=1\quad\alpha=2\quad k=5$		
	$MSE(\theta)$	$MSE(\alpha)$	$MSE(k)$	$MSE(\theta)$	$MSE(\alpha)$	$MSE(k)$
n=30	0.2584	$2.4364\ 10^{-3}$	$1.2763\ 10^{-2}$	$6.4092\ 10^{-2}$	$5.6487\ 10^{-4}$	$5.6327\ 10^{-4}$
n=50	0.0318	$7.2359\ 10^{-4}$	$4.8501\ 10^{-4}$	$1.2762\ 10^{-2}$	$2.7312\ 10^{-4}$	$1.7385\ 10^{-4}$
n=100	0.0110	$2.7785\ 10^{-4}$	1.065110^{-3}	$9.8506\ 10^{-3}$	$4.3718\ 10^{-5}$	$3.2146\ 10^{-3}$
	$\theta=1.5\quad\alpha=1\quad k=1.25$			$\theta=2\quad\alpha=3\quad k=1.25$		
	$MSE(\theta)$	$MSE(\alpha)$	$MSE(k)$	$MSE(\theta)$	$MSE(\alpha)$	$MSE(k)$
n=30	0.2546	$5.7258\ 10^{-4}$	1.093510^{-2}	0.2366	$2.05867\ 10^{-3}$	$1.6487\ 10^{-2}$
n=50	0.1581	$1.3610\ 10^{-4}$	$6.8303\ 10^{-3}$	0.0323	$1.5698\ 10^{-3}$	$6.2404\ 10^{-3}$
n=100	0.0401	$5.3779\ 10^{-5}$	$9.0554\ 10^{-5}$	0.0789	$3.7259\ 10^{-5}$	$1.9747\ 10^{-3}$
	$\theta=4\quad\alpha=3\quad k=3$			$\theta=1.5\quad\alpha=5\quad k=7$		
	$MSE(\theta)$	$MSE(\alpha)$	$MSE(k)$	$MSE(\theta)$	$MSE(\alpha)$	$MSE(k)$
n=30	2.0969	$8.8911\ 10^{-5}$	0.2603	0.1832	$6.3244\ 10^{-4}$	0.1130
n=50	0.5537	$5.0155\ 10^{-5}$	0.2510	$4.5248\ 10^{-2}$	$1.8568\ 10^{-4}$	$3.7919\ 10^{-3}$
n=100	0.3670	$1.1140\ 10^{-5}$	$2.0932\ 10^{-2}$	$2.4916\ 10^{-3}$	$4.7630\ 10^{-5}$	$5.3214\ 10^{-2}$
	$\theta=0.5\quad\alpha=0.3\quad k=0.9$			$\theta=1\quad\alpha=0.8\quad k=0.5$		
	$MSE(\theta)$	$MSE(\alpha)$	$MSE(k)$	$MSE(\theta)$	$MSE(\alpha)$	$MSE(k)$
n=30	0.1050	$4.4654\ 10^{-3}$	$9.8114\ 10^{-5}$	0.0388	$8.0237\ 10^{-3}$	$2.2877\ 10^{-4}$
n=50	0.0118	$1.5552\ 10^{-3}$	$1.8561\ 10^{-5}$	0.0209	$4.6262\ 10^{-3}$	$3.3000\ 10^{-5}$
n=100	0.0027	$3.4546\ 10^{-4}$	$2.1128\ 10^{-5}$	0.0027	$1.4167\ 10^{-3}$	$8.3529\ 10^{-5}$
	$\theta=0.75\quad\alpha=0.5\quad k=1.25$			$\theta=3\quad\alpha=1.5\quad k=2$		
	$MSE(\theta)$	$MSE(\alpha)$	$MSE(k)$	$MSE(\theta)$	$MSE(\alpha)$	$MSE(k)$
n=30	0.0240	$1.2623\ 10^{-3}$	$2.2451\ 10^{-4}$	2.3788	$9.3359\ 10^{-5}$	0.01964
n=50	0.0254	$6.6638\ 10^{-4}$	$1.4017\ 10^{-3}$	0.8301	$4.3801\ 10^{-5}$	$5.3191\ 10^{-3}$
n=100	0.0069	$1.6307\ 10^{-4}$	$2.9504\ 10^{-4}$	0.2535	$1.0024\ 10^{-5}$	$5.9578\ 10^{-4}$

Table 1 shows how the four biases vary with respect to n. Table 2 shows how the mean squared errors vary with respect to n. The mean squared errors for each parameter decrease to zero as $n\to\infty$. These numerical results coincide with the established theoretical results.

4 Application to real data sets

In this section, we give the applicability of LP distribution by considering three different data sets used by different researchers: Application to waiting times in a queue, Wheaton River Data, Application to bladder cancer patients, and compare them with different distribution, of which Lindley exponential, Lindley Weibull, Lindley, Power Lindley (see, Cooray, 2006), exponential Pareto, Pareto and gamma Lindley distributions. In each case, the parameters are estimated by maximum likelihood, as described in Section 6, using the R software.

In order to compare the above distributions with Lindley Pareto distribution, we consider criteria like $-2l$, AIC (Akaike information criterion), $AICC$ (corrected Akaike information criterion), BIC (Bayesian information criterion) and $HQIC$ (Hannan-Quinn information criterion) for the data set. The model selection is carried out using the following statistics:

$$AIC = -2LL + 2p, CAIC = -2LL + \frac{2pn}{n-p-1}$$

$$BIC = -2LL + p\log(n) \text{ and } HQIC = -2LL + 2p\log(\log(n))$$

For instance, it is well known that the AIC statistics favours models with large number of parameters in contrast to the Bayesian Information Criterion (BIC), which tends to present a better balance between the (negative) likelihood function and the number of parameters or model complexity.

Remark 3. Kolmogorov Smirnov test cannot be used in this case because the parameters are being estimated.

4.1 Illustration 1: Application to waiting times in a queue

We consider 100 observations on waiting time as a real example that happens before the customer received service in a bank. The data set represents the waiting time (mins) of one hundred (100) bank customers before service is being rendered. This data has previously been used by Ghitany et al. (2008a). Table 3 provides the estimated values of the model parameters. The information criterion values are given in Table 4.

Table 3. Parameter estimates for 100 bank customers

Distribution	Parameters
LP	$\hat{\theta}=0.1586 \quad \hat{\alpha}=0.801 \quad \hat{k}=1.0048$
LE	$\hat{\theta}=2.6501 \quad \hat{\lambda}=0.152$
EP	$\hat{k}=1.5137 \quad \hat{\alpha}=0.801 \quad \hat{\lambda}=0.0183$
GaL	$\hat{\theta}=0.2024 \quad \hat{\beta}=217.72$
L	$\hat{\theta}=0.187$
P	$\hat{\alpha}=0.801 \quad \hat{k}=0.4367$
LW	$\hat{\theta}=0.0003 \quad a=1.0096 \quad b=0.0014$
PL	$\hat{\theta}=0.153 \quad \hat{\alpha}=1.0832$

Table 4. The -LL, AIC, CAIC, BIC, HQIC for 100 bank customers

Distribution	-LL	AIC	CAIC	BIC	HQIC
LP	**308.9731**	**621.9462**	**622.0874**	**627.6346**	**623.9423**
LE	317.005	638.01	638.1337	643.2203	640.1187
EP	312.1154	628.2308	628.372	633.9192	630.2269
GaL	317.3066	638.6132	638.7369	643.8235	640.7219
L	319.00	640.00	640.0408	642.6052	641.0544
P	381.7586	765.5172	765.5637	767.9945	766.5153
LW	317.3267	640.6534	640.9034	648.4689	643.8165
PL	318.3186	640.6372	641.9156	645.8475	642.7459

4.2 Illustration 2: Wheaton River Data

In this subsection we illustrate the flexibility of the new distribution to model both heavy tailed and approximately symmetric data, which correspond to the exceedance of food peaks (in m^3/s) of the Wheaton river near Carcross in Yukon Territory (Canada) of 72 exceedance measures for the years 1958-1984. These data were analyzed by many authors (see for instance, Akinsete et al., 2008). We have chosen the same data in order to compare our results with other models proposed by these authors. Table 5 provides the estimated values. The -LL, AIC, CAIC, BIC and HQIC statistics for each model is provided in Table 6. It can be seen that our proposed distribution leads to a better fit than any of alternative approaches.

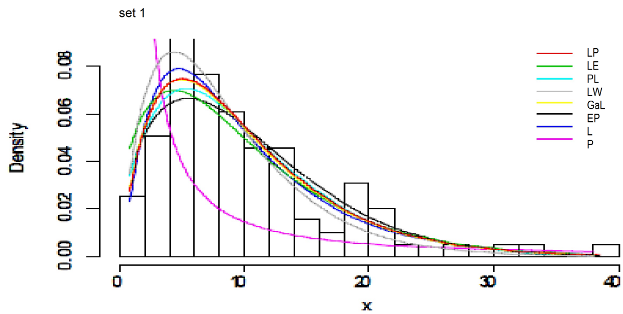

Figure 1: Estimated densities of the models for data set 1

Table 5. Parameter estimates for Wheaton river flood data

Distribution	Parameters		
LP	$\hat{\theta} = 0.1320$	$\hat{\alpha} = 0.1001$	$\hat{k} = 0.5921$
LE	$\hat{\theta} = 1.1210$	$\hat{\beta} = 0.0622$	
EP	$\hat{k} = 0.9320$	$\hat{\lambda} = 0.0115$	$\hat{\alpha} = 0.1001$
GaL	$\hat{\theta} = 0.0821$	$\hat{\beta} = 0.0760$	
L	$\hat{\theta} = 0.1531$		
P	$\hat{\alpha} = 0.1002$	$\hat{k} = 0.2405$	
WL	$\hat{\theta} = 0.0035$	$a = 0.5922$	$b = 0.0002$
PL	$\hat{\theta} = 0.3386$	$\hat{\alpha} = 0.7001$	

Table 6. The statistics -LL, AIC, CAIC, BIC, HQIC for Wheaton river flood data

Distribution	-LL	AIC	CAIC	BIC	HQIC
LP	249.3267	502.6534	502.7502	508.3418	504.9645
LE	251.5364	507.0728	507.1688	512.7769	509.3904
EP	249.3288	502.6576	502.7544	508.346	504.9687
GaL	252.128	508.256	508.352	513.9601	510.5736
L	264.2118	530.4236	530.4553	533.2756	531.5824
P	303.9486	609.8972	609.9292	612.7414	611.0528
LW	252.3039	510.6078	510.8013	519.1639	514.0842
PL	252.2218	508.4436	508.5396	514.1477	510.7612

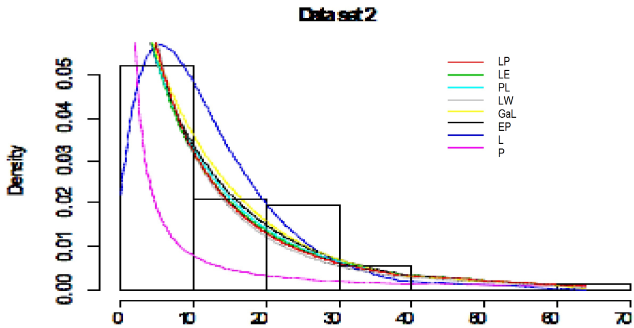

Figure 2: Estimated densities of the models for data set 2

4.3 Illustration 3: Application to bladder cancer patients

We consider a non-controlled data set corresponding to the remission times (in months) of a random sample of (128) bladder cancer patients. This cancer is a disease in which aberrant cells increase without control in the bladder and its application in survival analysis has been identified. The data set was given by Lee and Wang (2003). The results for these data are presented in Tables 7 and 8.

Table 7. Parameter estimates for bladder cancer data

Distribution	Parameters		
LP	$\hat{\theta} = 0.1229$	$\hat{\alpha} = 0.0801$	$\hat{k} = 0.6243$
LE	$\hat{\theta} = 1.2292$	$\hat{\lambda} = 0.0962$	
EP	$\hat{k} = 0.9379$	$\hat{\lambda} = 0.0128$	$\hat{\alpha} = 0.08$
GaL	$\hat{\theta} = 0.1167$	$\hat{\beta} = 0.1045$	
L	$\hat{\theta} = 0.1961$		
P	$\hat{\alpha} = 0.0801$	$\hat{k} = 0.2458$	
WL	$\hat{\theta} = 0.0027$	$a = 0.6316$	$b = 0.0002$
PL	$\hat{\theta} = 0.3855$	$\hat{\alpha} = 0.7443$	

Table 8. The statistics -LL, AIC, CAIC, BIC, HQIC for bladder cancer data

Distribution	-LL	AIC	CAIC	BIC	HQIC
LP	398.0184	800.0368	800.1336	805.7252	802.3479
LE	401.78	807.564	807.656	813.2641	809.8776
EP	400.3128	804.6256	804.7224	810.314	806.9367
GaL	402.9596	809.9192	810.0152	815.6233	812.2368
L	419.52	841.040	841.0717	843.892	842.1988
P	501.1292	1004.258	1004.29	1007.103	1005.414
WL	401.196	808.392	808.5855	816.9481	811.8684
PL	402.2373	808.4746	808.5706	814.1787	810.7922

According to Tables 4, 6, 8 and Figures 1, 2, 3, we can observe that LP distribution provide smallest -LL, AIC, CAIC, BIC and HQIC values as compared to Lindley exponential, Lindley Weibull, Lindley, Power Lindley, exponential Pareto, Pareto and gamma Lindley distributions, and hence best fits the data among all the models considered.

5 Conclusion

This work proposes more properties and simulations of the Lindley Pareto distribution generated by Lindley distribution. We investigate several of its structural properties such as an expansion for the density function and explicit expressions for the quantile function, maximum likelihood estimators of the parameters, mean deviation, and entropy. A simulation study is carried out to examine the bias and mean square error of the maximum likelihood estimators of the parameters. Several applications of the model to a real

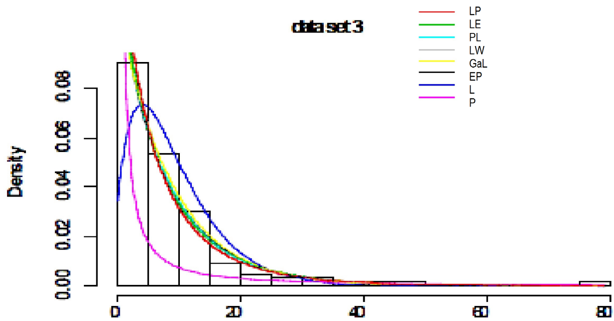

Figure 3: Estimated densities of the models for data set 3

data set are presented finally and compared with the fit attained by some other well-known one, two, three and four parameters. The adequacy of fits was assessed in terms AIC values, BIC values and density plots. We can show that the Lindley Pareto distribution can be used quite effectively in analyzing real lifetime data and actuarial science.

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APPENDIX

(1) Power Lindley distribution

$$f_1(x)=\frac{\alpha\theta^2}{(\theta+1)}(1+x^\alpha)x^{\alpha-1}\exp(-\theta x^\alpha)$$

(2)Lindley Weibull Distribution

$$f_5(x)=\frac{\alpha\theta^2}{b(\theta+1)}\left(\frac{x}{b}\right)^{\alpha-1}\left(1+\left(\frac{x}{b}\right)^\alpha\right)\exp\left(-\theta\left(\frac{x}{b}\right)^\alpha\right)$$

(3) Lindley Distribution

$$f_3(x)=\frac{\theta^2}{1+\theta}(1+x)\exp(-\theta x)$$

(4) Lindley Exponential distribution

$$f_4(x)=\frac{\lambda\theta^2\exp(-\lambda x)}{(\theta+1)}(1-\exp(-\lambda x))^{\theta-1}(1-\ln(1-\exp(-\lambda x)))$$

(5) Pareto Distribution

$$f_6(x)=k\frac{\alpha^k}{x^{k+1}}$$

(6) Exponential Pareto Distribution

$$f_6(x)=\frac{\lambda\alpha}{k}\left(\frac{x}{k}\right)^{\alpha-1}e^{-\lambda\left(\frac{x}{k}\right)^\alpha}$$

(7) Gamma Lindley distribution

$$f_2(x)=\frac{\theta^2((\beta+\beta\theta-\theta)x+1)e^{-\theta x}}{\beta(1+\theta)}$$

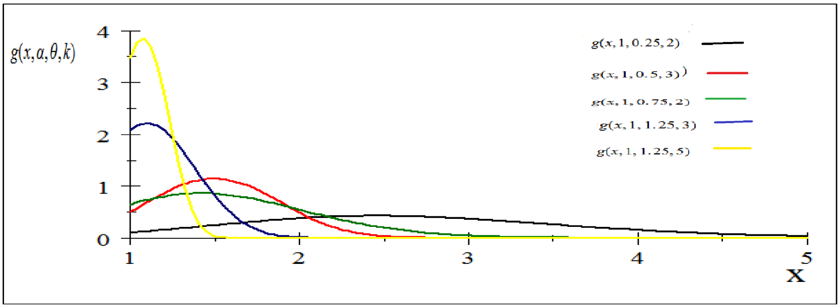

Figure 4: PDF plot for various values of parameters

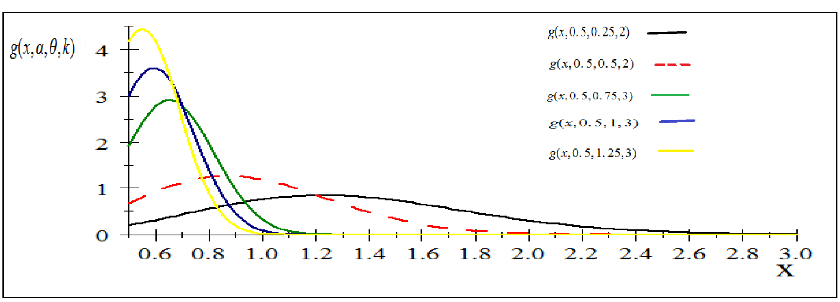

Figure 5: PDF plot for various values of parameters

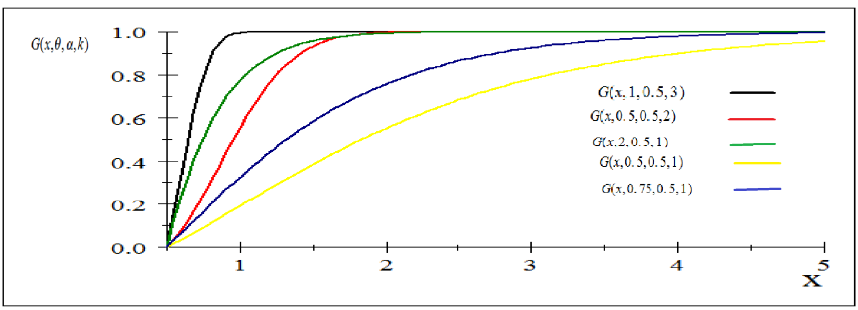

Figure 6: CDF plot for various values of parameters