STATISTICS IN TRANSITION new series, June 2017
Vol. 18, No. 2, pp. 291–310, DOI 10. 21307

A THREE-PARAMETER WEIGHTED LINDLEY DISTRIBUTION AND ITS APPLICATIONS TO MODEL SURVIVAL TIME

Rama Shanker[1], Kamlesh Kumar Shukla[2], Amarendra Mishra[3]

ABSTRACT

In this paper a three-parameter weighted Lindley distribution, including Lindley distribution introduced by Lindley (1958), a two-parameter gamma distribution, a two-parameter weighted Lindley distribution introduced by Ghitany et al. (2011) and exponential distribution as special cases, has been suggested for modelling lifetime data from engineering and biomedical sciences. The structural properties of the distribution including moments, coefficient of variation, skewness, kurtosis and index of dispersion have been derived and discussed. The reliability properties, including hazard rate function and mean residual life function, have been discussed. The estimation of its parameters has been discussed using the maximum likelihood method and the applications of the distribution have been explained through some survival time data of a group of patients suffering from head and neck cancer, and the fit has been compared with a one-parameter Lindley distribution and a two-parameter weighted Lindley distribution.

Key words: moments, stochastic ordering, hazard rate function, mean residual life function, maximum likelihood estimation, lifetime data, goodness of fit.

1. Introduction

The probability density function (p.d.f.) of the two-parameter weighted Lindley distribution (WLD), introduced by Ghitany et al. (2011) with parameters α and θ, is given by

$$f\left(x;\theta,\alpha\right)=\frac{\theta^{\alpha+1}}{\left(\theta+\alpha\right)}\frac{x^{\alpha-1}}{\Gamma\left(\alpha\right)}\left(1+x\right)e^{-\theta x}\ ;x>0,\ \theta>0,\ \alpha>0 \qquad (1.1)$$

[1] Department of Statistics, Eritrea Institute of Technology, Asmara, Eritrea. E-mail: shnakerrama2009@gmail.com.
[2] Department of Statistics, Eritrea Institute of Technology, Asmara, Eritrea. E-mail: kkshukla22@gmail.com.
[3] Department of Statistics, Patna University, Patna, India. E-mail: mishraamar@rediffmail.com.

where

$$\Gamma(\alpha)=\int_0^{\infty} e^{-y}\, y^{\alpha-1}\, dy;\ \alpha>0$$

is the complete gamma function. Its structural properties including moments, hazard rate function, mean residual life function, estimation of parameters and applications to modelling survival time data have been discussed by Ghitany et al. (2011). The corresponding cumulative distribution function (c.d.f.) of WLD (1.1) can be obtained as

$$F(x;\theta,\alpha)=1-\frac{(\theta+\alpha)\Gamma(\alpha,\theta x)+(\theta x)^{\alpha}\, e^{-\theta x}}{(\theta+\alpha)\Gamma(\alpha)}\ ;x>0,\ \theta>0,\ \alpha>0 \quad (1.2)$$

where

$$\Gamma(\alpha,z)=\int_z^{\infty} e^{-y}\, y^{\alpha-1}\, dy;\alpha>0,\ z\geq 0 \quad (1.3)$$

is the upper incomplete gamma function.

It can be easily shown that at $\alpha=1$, WLD (1.1) reduces to Lindley (1958) distribution having p.d.f.

$$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\quad ;x>0,\ \theta>0 \quad (1.4)$$

It can be easily verified that the p.d.f. (1.4) is a two-component mixture of exponential (θ) and gamma $(2,\theta)$ distributions. Ghitany et al. (2008) have conducted a detailed study about various properties of Lindley distribution including skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, stress-strength reliability, among other things; estimation of its parameter and application to model waiting time data in a bank. Shanker and Mishra (2013 a, 2013 b), Shanker and Amanuel (2013), and Shanker *et al.* (2013) have obtained different forms of the two-parameter Lindley distribution and discussed their various properties including skewness, kurtosis, index of dispersion, hazard rate function, mean residual life function, stochastic ordering, mean deviation, stress-strength reliability; estimation of parameters and their applications to model waiting and survival times data. Sankaran (1970) has obtained discrete Poisson-Lindley distribution by mixing Poisson distribution with Lindley (1958) distribution and studied its properties based on moments, estimation of parameters and applications to model count data from biological sciences. Shanker *et al.* (2015) have discussed a comparative study of Lindley and exponential distributions for modelling various lifetime data sets from biomedical science and engineering, and concluded that there are lifetime data where exponential distribution gives better fit than Lindley distribution and in majority of data sets Lindley distribution gives better fit than exponential distribution.

Further, p.d.f. (1.1) can also be expressed as a two-component mixture of gamma (α,θ) and gamma $(\alpha+1,\theta)$ distributions. We have

$$f(x;\theta,\alpha)=p\, f_1(x;\theta,\alpha)+(1-p)\, f_2(x;\theta,\alpha+1), \tag{1.5}$$

where

$$p=\frac{\theta}{\theta+\alpha},\ f_1(x;\theta,\alpha)=\frac{\theta^{\alpha}}{\Gamma(\alpha)}\, e^{-\theta x}x^{\alpha-1}, \text{ and}$$

$$f_2(x;\theta,\alpha+1)=\frac{\theta^{\alpha+1}}{\Gamma(\alpha+1)}\, e^{-\theta x}x^{\alpha+1-1}.$$

Ghitany *et al.* (2011) have discussed the structural properties of WLD including the nature of its p.d.f., hazard rate function, mean residual life function and applications to survival data using maximum likelihood estimation. It has been shown by Ghitany *et al.* (2011) that the shapes of hazard rate function and mean residual life function are decreasing, increasing and bathtub and thus has the potential to model survival time data of different nature. Shanker *et al.* (2016) have discussed some of its important statistical and mathematical properties including central moments, coefficient of variation, skewness, kurtosis, index of dispersion, stochastic ordering and the applications to modelling lifetime data from engineering and biomedical sciences.

In the present paper, a three-parameter weighted Lindley distribution, which includes Lindley (1958) distribution, WLD introduced by Ghitany et al. (2011), two-parameter gamma distribution and exponential distribution as particular cases, has been proposed and discussed. Its moments about origin and central moments, coefficient of variation, skewness, kurtosis and index of dispersion have been derived. The hazard rate function and the mean residual life function of the distribution have been derived and their shapes have been discussed for varying values of the parameters. The estimation of its parameters has been discussed using maximum likelihood method. Finally, the goodness of fit and the applications of the distribution have been explained through some survival data and the fit has been compared with a one-parameter Lindley distribution and the two-parameter WLD.

2. A three-parameter weighted Lindley distribution

A three-parameter weighted Lindley distribution (TPWLD) having parameters θ,α, and β can be defined by the probability density function

$$f(x;\theta,\alpha,\beta)=\frac{\theta^{\alpha+1}}{(\beta\theta+\alpha)}\frac{x^{\alpha-1}}{\Gamma(\alpha)}(\beta+x)e^{-\theta x}\quad ;x>0,\theta>0,\alpha>0,\beta\theta+\alpha>0 \tag{2.1}$$

where α and β are shape parameters and θ is a scale parameter.

It can be easily verified that the Lindley distribution introduced by Lindley (1958) and the two-parameter WLD introduced by Ghitany et al. (2011) are particular cases of (2.1) for $\alpha=\beta=1$ and $\beta=1$ respectively. A two-parameter gamma$\left(\alpha,\theta\right)$ distribution is a particular case of TPWLD for $\beta\to\infty$. Again, for $\alpha=1$ and $\beta\to\infty$, TPWLD reduces to the one-parameter exponential distribution. Further, the p.d.f. (2.1) can be easily expressed as a two-component mixture of gamma $\left(\alpha,\theta\right)$ and gamma $\left(\alpha+1,\theta\right)$ distributions. We have

$$f\left(x;\theta,\alpha,\beta\right)=p\,f_1\left(x;\theta,\alpha\right)+\left(1-p\right)f_2\left(x;\theta,\alpha+1\right), \qquad (2.2)$$

where

$$p=\frac{\beta\theta}{\beta\theta+\alpha},\quad f_1\left(x;\theta,\alpha\right)=\frac{\theta^{\alpha}}{\Gamma\left(\alpha\right)}\,e^{-\theta x}x^{\alpha-1},$$

$$f_2\left(x;\theta,\alpha+1\right)=\frac{\theta^{\alpha+1}}{\Gamma\left(\alpha+1\right)}\,e^{-\theta x}x^{\alpha+1-1}.$$

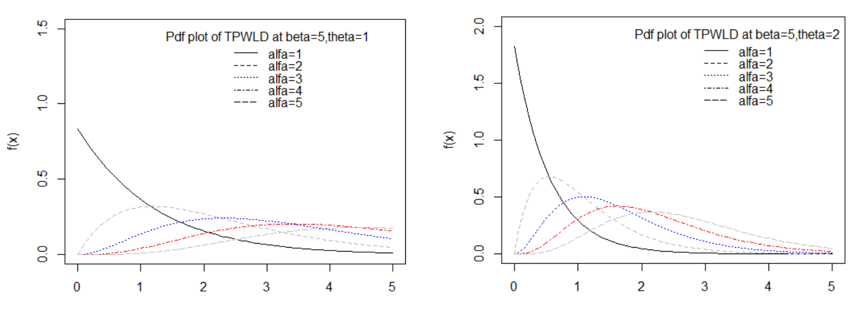
The corresponding cumulative distribution function of TPWLD (2.1) can be obtained as

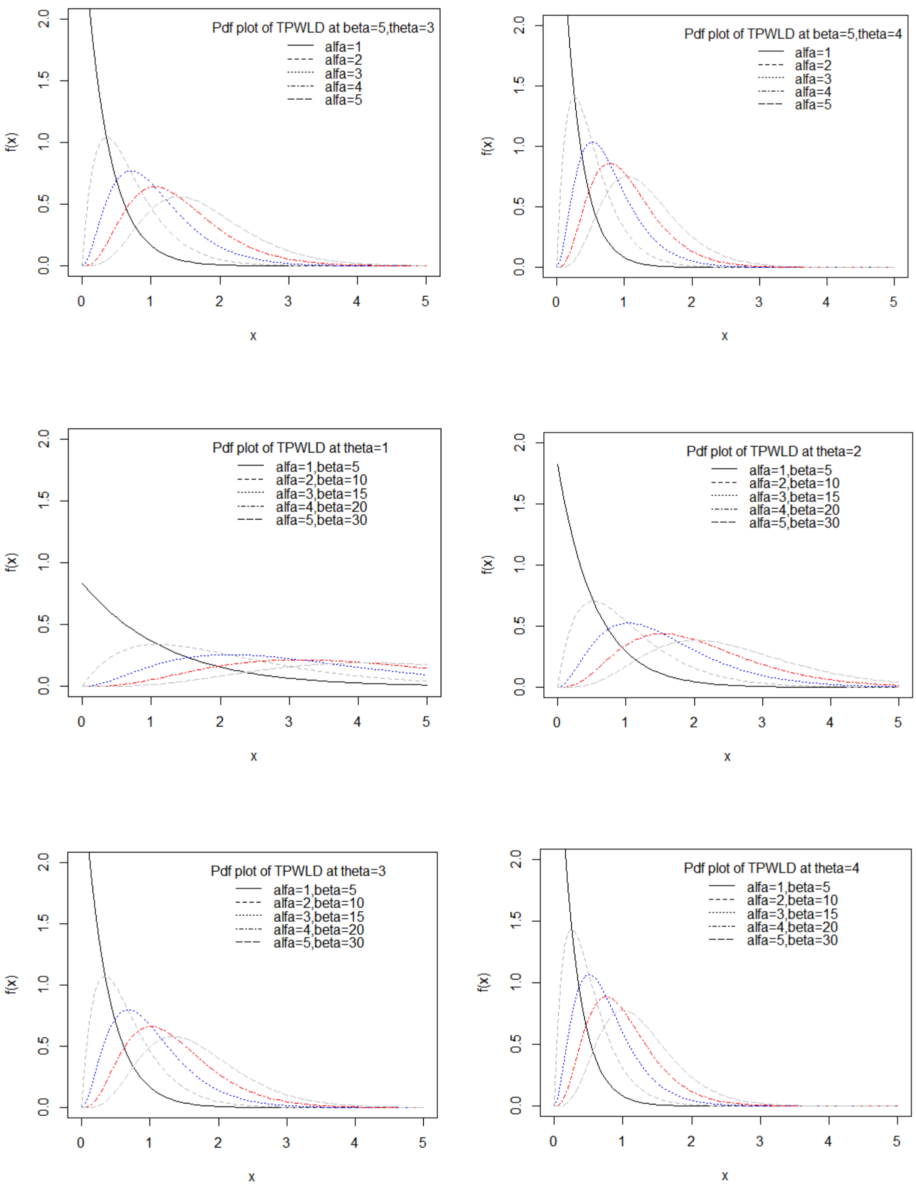
$$F\left(x;\theta,\alpha\right)=1-\frac{\left(\beta\theta+\alpha\right)\Gamma\left(\alpha,\theta x\right)+\left(\theta x\right)^{\alpha}e^{-\theta x}}{\left(\beta\theta+\alpha\right)\Gamma\left(\alpha\right)};$$

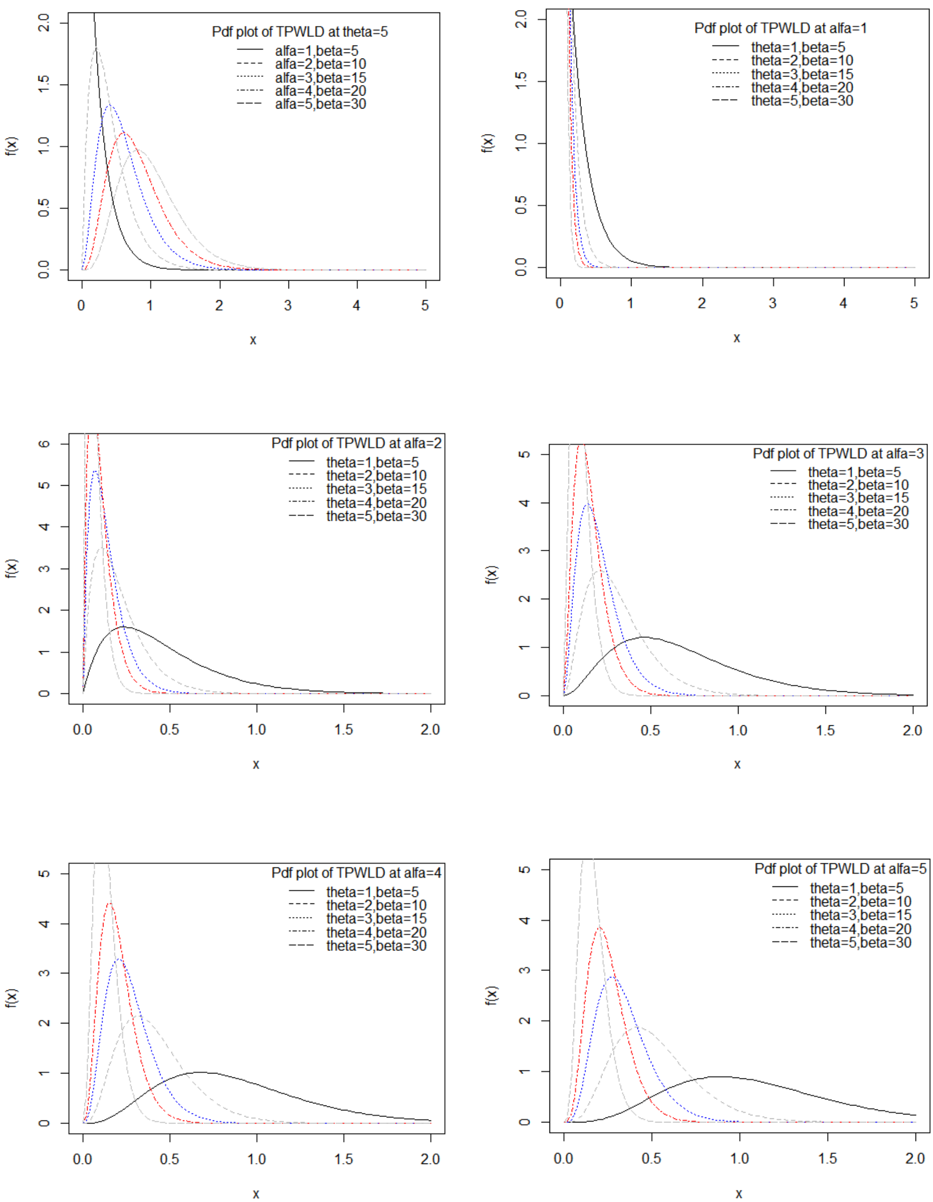
$$x>0,\theta>0,\alpha>0,\beta\theta+\alpha>0 \qquad (2.3)$$

where $\Gamma\left(\alpha,z\right)$ is the upper incomplete gamma function defined in (1.3).

The nature of the p.d.f. of TPWLD for varying values of the parameters has been shown graphically in Figure 1.




Pdf plot of TPWLD at beta=5,theta=3
alfa=1
alfa=2
alfa=3
alfa=4
alfa=5
f(x)
x
Pdf plot of TPWLD at beta=5,theta=4
alfa=1
alfa=2
alfa=3
alfa=4
alfa=5
f(x)
x
Pdf plot of TPWLD at theta=1
alfa=1,beta=5
alfa=2,beta=10
alfa=3,beta=15
alfa=4,beta=20
alfa=5,beta=30
f(x)
x
Pdf plot of TPWLD at theta=2
alfa=1,beta=5
alfa=2,beta=10
alfa=3,beta=15
alfa=4,beta=20
alfa=5,beta=30
f(x)
x
Pdf plot of TPWLD at theta=3
alfa=1,beta=5
alfa=2,beta=10
alfa=3,beta=15
alfa=4,beta=20
alfa=5,beta=30
f(x)
x
Pdf plot of TPWLD at theta=4
alfa=1,beta=5
alfa=2,beta=10
alfa=3,beta=15
alfa=4,beta=20
alfa=5,beta=30
f(x)
x


Pdf plot of TPWLD at theta=5
alfa=1,beta=5
alfa=2,beta=10
alfa=3,beta=15
alfa=4,beta=20
alfa=5,beta=30
f(x)
x
Pdf plot of TPWLD at alfa=1
theta=1,beta=5
theta=2,beta=10
theta=3,beta=15
theta=4,beta=20
theta=5,beta=30
f(x)
x
Pdf plot of TPWLD at alfa=2
theta=1,beta=5
theta=2,beta=10
theta=3,beta=15
theta=4,beta=20
theta=5,beta=30
f(x)
x
Pdf plot of TPWLD at alfa=3
theta=1,beta=5
theta=2,beta=10
theta=3,beta=15
theta=4,beta=20
theta=5,beta=30
f(x)
x
Pdf plot of TPWLD at alfa=4
theta=1,beta=5
theta=2,beta=10
theta=3,beta=15
theta=4,beta=20
theta=5,beta=30
f(x)
x
Pdf plot of TPWLD at alfa=5
theta=1,beta=5
theta=2,beta=10
theta=3,beta=15
theta=4,beta=20
theta=5,beta=30
f(x)
x

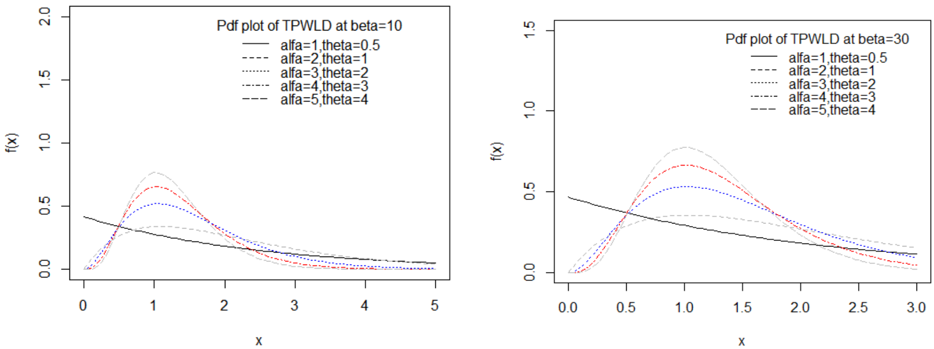


Figure 1. Nature of the p.d.f. of TPWLD for varying values of the parameters

3. Moments and related measures

Using the mixture representation (2.2), the rth moment about origin of TPWLD (2.1) can be obtained as

$$\mu_r' = E\left(X^r\right) = p\int_0^\infty x^r f_1\left(x;\theta,\alpha\right)dx + \left(1-p\right)\int_0^\infty x^r f_2\left(x;\theta,\alpha+1\right)dx$$

$$= \frac{\left[\beta\theta+\alpha+r\right]\Gamma\left(\alpha+r\right)}{\theta^r\left(\beta\theta+\alpha\right)\Gamma\left(\alpha\right)};\ r=1,2,3,... \qquad (3.1)$$

Substituting $r=1,2,3,$ and 4 in (3.1), the first four moments about origin of TPWLD are obtained as

$$\mu_1' = \frac{\alpha\left(\beta\theta+\alpha+1\right)}{\theta\left(\beta\theta+\alpha\right)}$$

$$\mu_2' = \frac{\alpha\left(\alpha+1\right)\left(\beta\theta+\alpha+2\right)}{\theta^2\left(\beta\theta+\alpha\right)}$$

$$\mu_3' = \frac{\alpha\left(\alpha+1\right)\left(\alpha+2\right)\left(\beta\theta+\alpha+3\right)}{\theta^3\left(\beta\theta+\alpha\right)}$$

$$\mu_4' = \frac{\alpha\left(\alpha+1\right)\left(\alpha+2\right)\left(\alpha+3\right)\left(\beta\theta+\alpha+4\right)}{\theta^4\left(\beta\theta+\alpha\right)}$$

It can be easily verified that these raw moments reduce to the corresponding raw moments of a two-parameter gamma distribution for $\beta\to\infty$.

Again, using the relationship between central moments and moments about origin, the central moments of TPWLD are obtained as

$$\mu_2=\frac{\alpha\left[\beta^2\theta^2+2\beta\theta(\alpha+1)+\alpha(\alpha+1)\right]}{\theta^2(\beta\theta+\alpha)^2}$$

$$\mu_3=\frac{2\alpha\left[\beta^3\theta^3+3\beta^2\theta^2(\alpha+1)+3\beta\theta\alpha(\alpha+1)+\alpha^2(\alpha+1)\right]}{\theta^3(\beta\theta+\alpha)^3}$$

$$\mu_4=\frac{3\alpha\begin{bmatrix}(\alpha+1)\beta^4\theta^4+4(\alpha^2+3\alpha+2)\beta^3\theta^3+2\alpha(3\alpha^2+11\alpha+8)\beta^2\theta^2\\+4\alpha^2(\alpha^2+4\alpha+3)\beta\theta+\alpha^3(\alpha^2+4\alpha+3)\end{bmatrix}}{\theta^4(\beta\theta+\alpha)^4}$$

The expressions for coefficient of variation (C.V.), coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis (β_2), and index of dispersion (γ) of TPWLD (2.1) are thus obtained as

$$C.V.=\frac{\sigma}{\mu_1'}=\frac{\sqrt{\alpha\left[\beta^2\theta^2+2\beta\theta(\alpha+1)+\alpha(\alpha+1)\right]}}{\alpha(\beta\theta+\alpha+1)}$$

$$\sqrt{\beta_1}=\frac{\mu_3}{\mu_2^{3/2}}=\frac{2\alpha\left[\beta^3\theta^3+3\beta^2\theta^2(\alpha+1)+3\beta\theta\alpha(\alpha+1)+\alpha^2(\alpha+1)\right]}{\left[\alpha\left\{\beta^2\theta^2+2\beta\theta(\alpha+1)+\alpha(\alpha+1)\right\}\right]^{3/2}}$$

$$\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{3\begin{bmatrix}(\alpha+1)\beta^4\theta^4+4(\alpha^2+3\alpha+2)\beta^3\theta^3+2\alpha(3\alpha^2+11\alpha+8)\beta^2\theta^2\\+4\alpha^2(\alpha^2+4\alpha+3)\beta\theta+\alpha^3(\alpha^2+4\alpha+3)\end{bmatrix}}{\alpha\left\{\beta^2\theta^2+2\beta\theta(\alpha+1)+\alpha(\alpha+1)\right\}^2}$$

$$\gamma=\frac{\sigma^2}{\mu_1'}=\frac{\left[\beta^2\theta^2+2\beta\theta(\alpha+1)+\alpha(\alpha+1)\right]}{\theta(\beta\theta+\alpha)(\beta\theta+\alpha+1)}$$

The nature of the coefficient of variation, coefficient of skewness, coefficient of kurtosis and index of dispersion of TPWLD for varying values of the parameters have been shown in Figure 2.

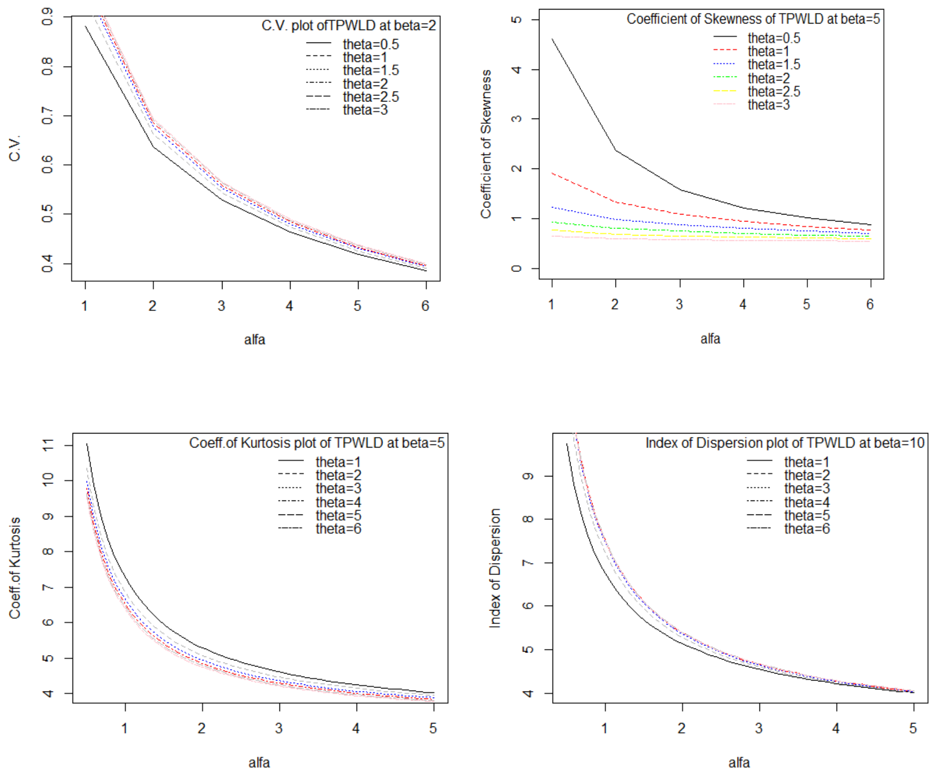


Figure 2. Graphs of coefficient of variation, coefficient of skewness, coefficient of kurtosis and index of dispersion of TPWLD for varying values of the parameters

4. Stochastic ordering

The stochastic ordering of positive continuous random variables is an important tool for judging their comparative behaviour. A continuous random variable X is said to be smaller than a continuous random variable Y in the

(i) stochastic order $\left(X \leq_{st} Y\right)$ if $F_X\left(x\right) \geq F_Y\left(x\right)$ for all x

(ii) hazard rate order $\left(X \leq_{hr} Y\right)$ if $h_X\left(x\right) \geq h_Y\left(x\right)$ for all x

(iii) mean residual life order $\left(X \leq_{mrl} Y\right)$ if $m_X\left(x\right) \leq m_Y\left(x\right)$ for all x

(iv) likelihood ratio order $\left(X \leq_{lr} Y\right)$ if $\dfrac{f_X\left(x\right)}{f_Y\left(x\right)}$ decreases in x.

The following stochastic ordering relationships due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of continuous distributions

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y$$
$$\Downarrow$$
$$X \le_{st} Y$$

TPWLD is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem:

Theorem: Let X ~ TPWLD $(\theta_1, \alpha_1, \beta_1)$ and Y ~ TPWLD$(\theta_2, \alpha_2, \beta_2)$. Then, the following results hold true

(i) If $\alpha_1 = \alpha_2, \beta_1 = \beta_2$ and $\theta_1 > \theta_2$, then $X \le_{lr} Y$, $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

(ii) If $\alpha_1 < \alpha_2, \beta_1 = \beta_2$ and $\theta_1 = \theta_2$, then $X \le_{lr} Y$, $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

(iii) If $\alpha_1 = \alpha_2, \beta_1 > \beta_2$ and $\theta_1 = \theta_2$, then $X \le_{lr} Y$, $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

(iv) If $\alpha_1 < \alpha_2, \beta_1 > \beta_2$ and $\theta_1 > \theta_2$, then $X \le_{lr} Y$, $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Proof: We have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1^{\alpha_1+1}(\beta_2\theta_2+\alpha_2)\Gamma(\alpha_2)}{\theta_2^{\alpha_2+1}(\beta_1\theta_1+\alpha_1)\Gamma(\alpha_1)} x^{\alpha_1-\alpha_2}\left(\frac{\beta_1+x}{\beta_2+x}\right)e^{-(\theta_1-\theta_2)x} \quad ; x>0$$

Now

$$\log\frac{f_X(x)}{f_Y(x)} = \log\left[\frac{\theta_1^{\alpha_1+1}(\beta_2\theta_2+\alpha_2)\Gamma(\alpha_2)}{\theta_2^{\alpha_2+1}(\beta_1\theta_1+\alpha_1)\Gamma(\alpha_1)}\right] + (\alpha_1-\alpha_2)\log x + \log\left(\frac{\beta_1+x}{\beta_2+x}\right) - (\theta_1-\theta_2)x \cdot$$

This gives $$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1-\alpha_2}{x} + \frac{\beta_2-\beta_1}{(\beta_1+x)(\beta_2+x)} - (\theta_1-\theta_2).$$

Thus for $\alpha_1 = \alpha_2, \beta_1 = \beta_2$ and $\theta_1 > \theta_2$, $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} < 0$. This means that $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$, and thus (i) is verified. Similarly, (ii), (iii) and (iv) can easily be verified.

5. Hazard rate function and mean residual life function

5.1. Hazard rate function

Using the mixture representation (2.2), the survival (reliability) function of TPWLD can be obtained as

$$S(x) = P(X > x) = p\int_x^\infty f_1(y;\theta,\alpha)\,dy + (1-p)\int_x^\infty f_2(y;\theta,\alpha+1)\,dy$$

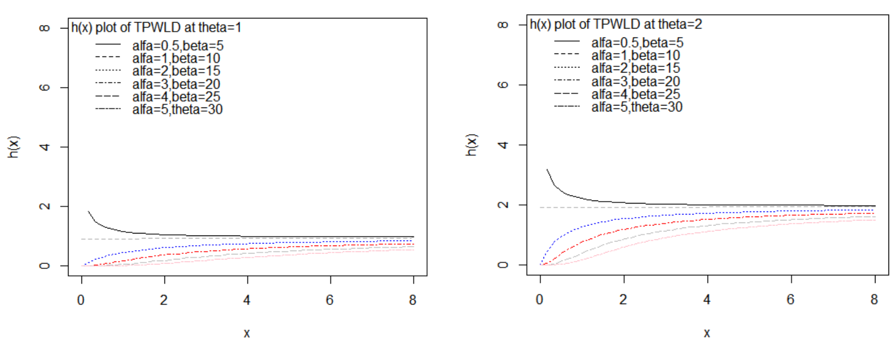
$$= \frac{(\beta\theta+\alpha)\Gamma(\alpha,\theta x) + (\theta x)^\alpha e^{-\theta x}}{(\beta\theta+\alpha)\Gamma(\alpha)} \quad (5.1.1)$$

where $\Gamma(\alpha, z)$ is the upper incomplete gamma function defined in (1.3).

The hazard (or failure) rate function, $h(x)$ of TPWLD is thus obtained as

$$h(x) = \frac{f(x)}{S(x)} = \frac{\theta^{\alpha+1}x^{\alpha-1}(\beta+x)e^{-\theta x}}{(\beta\theta+\alpha)\Gamma(\alpha,\theta x)+(\theta x)^\alpha e^{-\theta x}} \;; x>0, \theta>0, \alpha>0, \beta\theta+\alpha>0 \quad (5.1.2)$$

The shapes of the hazard rate function, $h(x)$ of TPWLD for varying values of the parameters are shown in Figure 3.



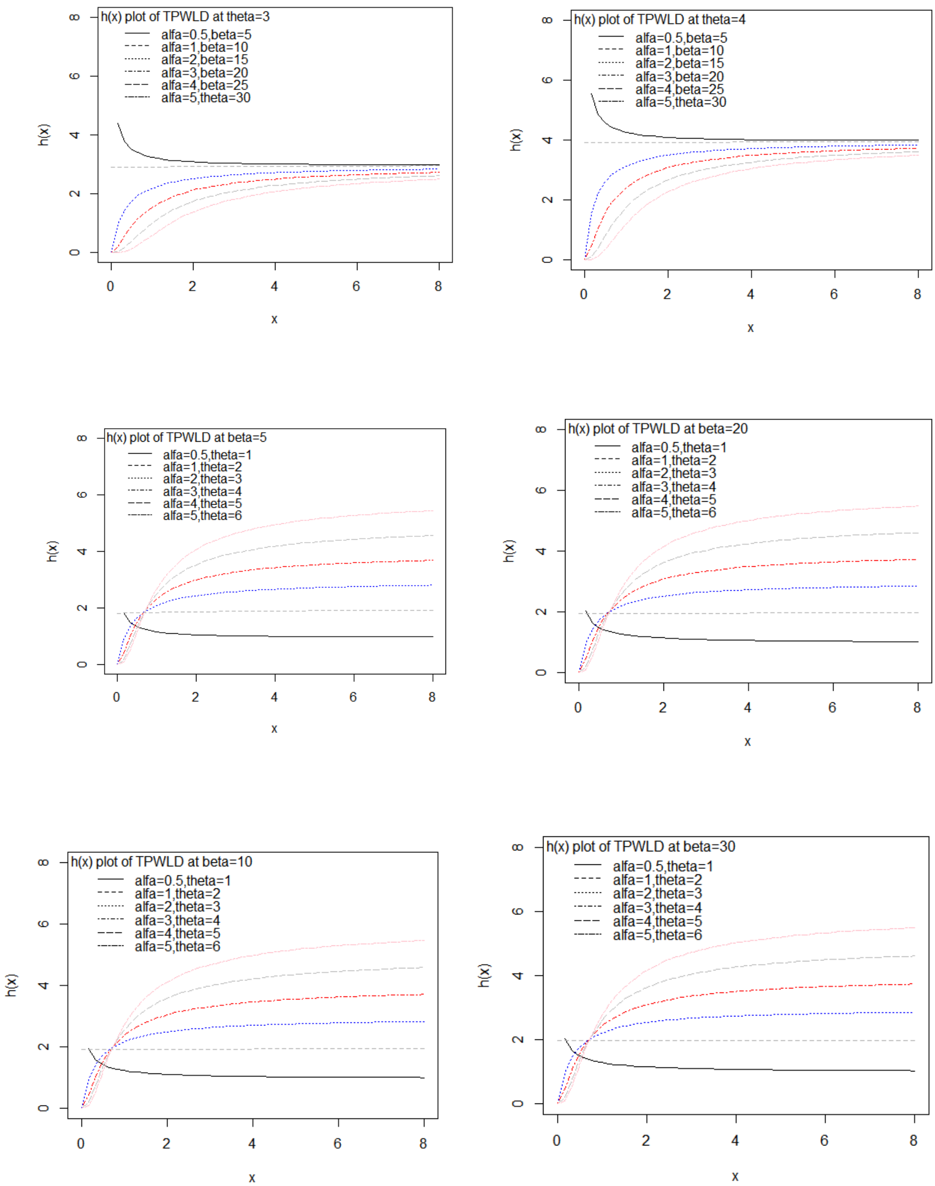


Figure 3. Graphs of the hazard rate function, $h(x)$ of TPWLD for varying values of the parameters

5.2. Mean residual life function

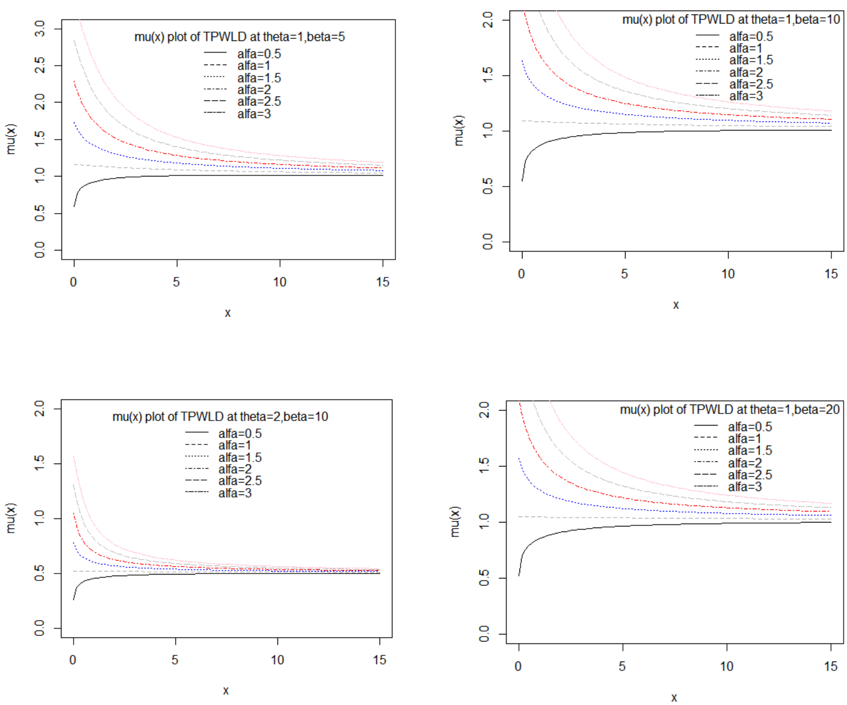
Using the mixture representation (2.2), the mean residual life function $m(x) = E(X - x \mid X > x)$ of TPWLD can be obtained as

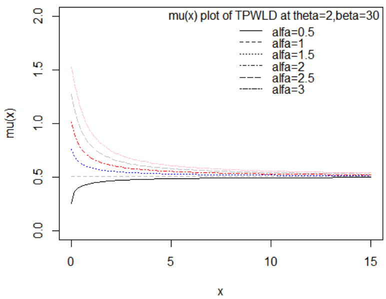
$$m(x) = \frac{1}{S(x)} \int_x^\infty y\, f(y)\, dy - x$$

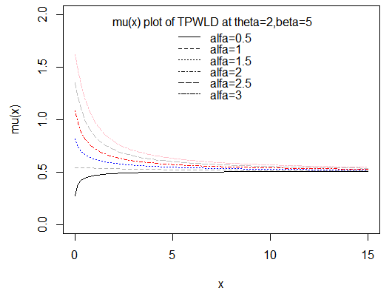
$$= \frac{1}{S(x)} \left[p \int_x^\infty y\, f_1(y;\theta,\alpha)\, dy + (1-p) \int_x^\infty y\, f_2(y;\theta,\alpha+1)\, dy \right] - x$$

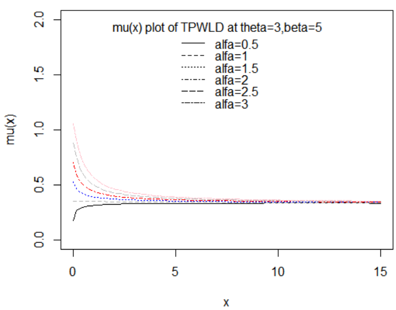
$$= \frac{\left[\alpha(\beta\theta+\alpha+1) - (\beta\theta+\alpha)(\theta x)\right]\Gamma(\alpha,\theta x) + (\beta\theta+\alpha+1)(\theta x)^{\alpha} e^{-\theta x}}{\theta\left[(\beta\theta+\alpha)\Gamma(\alpha,\theta x) + (\theta x)^{\alpha} e^{-\theta x}\right]}$$

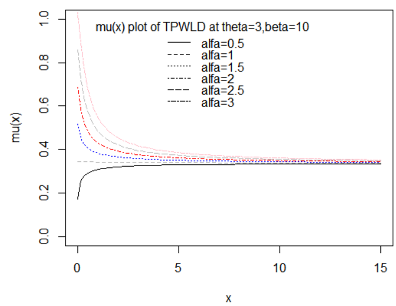
The shapes of the mean residual life function, $m(x)$ of TPWLD for varying values of the parameters are shown in Figure 4.

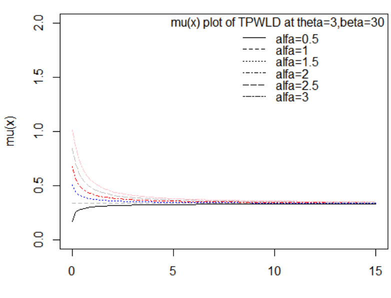


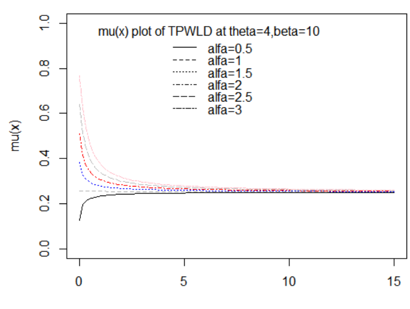

mu(x) plot of TPWLD at theta=2,beta=30
alfa=0.5
alfa=1
alfa=1.5
alfa=2
alfa=2.5
alfa=3
mu(x)
x


mu(x) plot of TPWLD at theta=2,beta=5
alfa=0.5
alfa=1
alfa=1.5
alfa=2
alfa=2.5
alfa=3
mu(x)
x


mu(x) plot of TPWLD at theta=3,beta=5
alfa=0.5
alfa=1
alfa=1.5
alfa=2
alfa=2.5
alfa=3
mu(x)
x


mu(x) plot of TPWLD at theta=3,beta=10
alfa=0.5
alfa=1
alfa=1.5
alfa=2
alfa=2.5
alfa=3
mu(x)
x


mu(x) plot of TPWLD at theta=3,beta=30
alfa=0.5
alfa=1
alfa=1.5
alfa=2
alfa=2.5
alfa=3
mu(x)
x


mu(x) plot of TPWLD at theta=4,beta=10
alfa=0.5
alfa=1
alfa=1.5
alfa=2
alfa=2.5
alfa=3
mu(x)
x

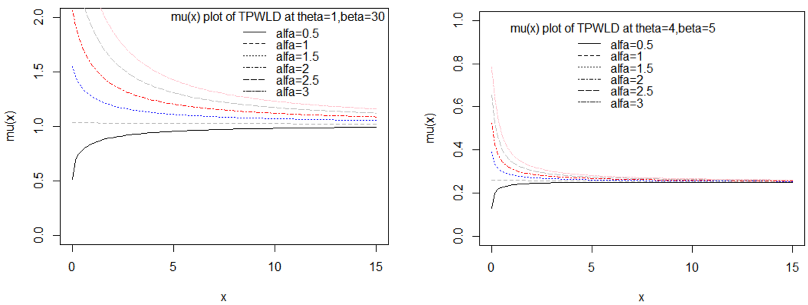


Figure 4. Graphs of the mean residual life function, $m(x)$ of TPWLD for varying values of the parameters

6. Maximum likelihood estimation

Let $(x_1, x_2, x_3, \dots, x_n)$ be a random sample of size n from TPWLD (2.1). The likelihood function, L of TPWLD is given by

$$L = \left(\frac{\theta^{\alpha+1}}{\beta\theta+\alpha}\right)^n \frac{1}{(\Gamma(\alpha))^n} \prod_{i=1}^{n} x_i^{\alpha-1}(\beta+x_i)e^{-n\theta\bar{x}}\text{ ; } \bar{x} \text{ being the sample}$$

mean.

The natural log likelihood function is thus obtained as

$$\ln L = n\left[(\alpha+1)\ln\theta - \ln(\beta\theta+\alpha) - \ln(\Gamma(\alpha))\right] + (\alpha-1)\sum_{i=1}^{n}\ln(x_i) + \sum_{i=1}^{n}\ln(\beta+x_i) - n\theta\bar{x}$$

The maximum likelihood estimates (MLEs) $(\hat{\theta}, \hat{\alpha}, \hat{\beta})$ of parameters (θ, α, β) of TPWLD are the solution of the following nonlinear equations

$$\frac{\partial \ln L}{\partial\theta} = \frac{n(\alpha+1)}{\theta} - \frac{n\beta}{\beta\theta+\alpha} - n\bar{x} = 0$$

$$\frac{\partial \ln L}{\partial\alpha} = n\ln\theta - \frac{n}{\beta\theta+\alpha} - n\psi(\alpha) + \sum_{i=1}^{n}\ln(x_i) = 0$$

$$\frac{\partial \ln L}{\partial\beta} = \frac{-n\theta}{\beta\theta+\alpha} + \sum_{i=1}^{n}\frac{1}{\beta+x_i} = 0$$

where $\bar{x}$ is the sample mean and $\psi(\alpha)=\frac{d}{d\alpha}\ln\Gamma(\alpha)$ is the digamma function. It should be noted that the first equation gives the expression for the mean as $\bar{x}=\frac{\alpha(\beta\theta+\alpha+1)}{\theta(\beta\theta+\alpha)}$.

These three log likelihood equations do not seem to be solved directly. However, Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 \ln L}{\partial\theta^2}=\frac{-n(\alpha+1)}{\theta^2}+\frac{n\beta^2}{(\beta\theta+\alpha)^2}$$

$$\frac{\partial^2 \ln L}{\partial\alpha^2}=\frac{n}{(\beta\theta+\alpha)^2}-n\psi'(\alpha)$$

$$\frac{\partial^2 \ln L}{\partial\beta^2}=\frac{n\theta^2}{(\beta\theta+\alpha)^2}-\sum_{i=1}^{n}\frac{1}{(\beta+x_i)^2}$$

$$\frac{\partial^2 \ln L}{\partial\theta\,\partial\alpha}=\frac{n}{\theta}+\frac{n\beta}{(\beta\theta+\alpha)^2}$$

$$\frac{\partial^2 \ln L}{\partial\theta\,\partial\beta}=\frac{-n\alpha}{(\beta\theta+\alpha)^2}$$

$$\frac{\partial^2 \ln L}{\partial\alpha\,\partial\beta}=\frac{n\theta}{(\beta\theta+\alpha)^2}$$

where $\psi'(\alpha)=\frac{d}{d\alpha}\psi(\alpha)$ is the tri-gamma function.

For the maximum likelihood estimates $(\hat{\theta},\hat{\alpha},\hat{\beta})$ of (θ,α,β) of TPWLD (2.1), the following equations can be solved

$$\begin{bmatrix}\frac{\partial^2 \ln L}{\partial\theta^2} & \frac{\partial^2 \ln L}{\partial\theta\,\partial\alpha} & \frac{\partial^2 \ln L}{\partial\theta\,\partial\beta}\\ \frac{\partial^2 \ln L}{\partial\theta\,\partial\alpha} & \frac{\partial^2 \ln L}{\partial\alpha^2} & \frac{\partial^2 \ln L}{\partial\alpha\,\partial\beta}\\ \frac{\partial^2 \ln L}{\partial\theta\,\partial\beta} & \frac{\partial^2 \ln L}{\partial\alpha\,\partial\beta} & \frac{\partial^2 \ln L}{\partial\beta^2}\end{bmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\alpha}=\alpha_0\\ \hat{\beta}=\beta_0}}\begin{bmatrix}\hat{\theta}-\theta_0\\ \hat{\alpha}-\alpha_0\\ \hat{\beta}-\beta_0\end{bmatrix}=\begin{bmatrix}\frac{\partial \ln L}{\partial\theta}\\ \frac{\partial \ln L}{\partial\alpha}\\ \frac{\partial \ln L}{\partial\beta}\end{bmatrix}$$

where $(\theta_0, \alpha_0, \beta_0)$ are the initial values of (θ, α, β) respectively. These equations are solved iteratively using any numerical iterative methods until sufficiently close values of $(\hat{\theta}, \hat{\alpha}, \hat{\beta})$ are obtained.

7. Applications and goodness of fit

In this section, the applications and goodness of fit of TPWLD has been discussed for several lifetime data and the fit is compared with a one-parameter Lindley distribution and a two-parameter WLD. In order to compare TPWLD, WLD and Lindley distribution, $-2\ln L$, AIC (Akaike information criterion), K-S Statistic (Kolmogorov-Smirnov Statistic) and p-value for two data sets have been computed and presented in Table 1. The formulae for AIC and K-S Statistics are as follows: $AIC = -2\log L + 2k$, and $K\text{-}S = \sup_x \left| F_n(x) - F_0(x) \right|$, where k is the number of parameters involved in the respective distributions, n is the sample size and $F_n(x)$ is the empirical distribution function. The best distribution corresponds to the lower values of $-2\ln L$, AIC and K-S statistic and higher p-value.

The goodness of fit of TPWLD, WLD, and Lindley distribution for data set 1 and 2 are based on maximum likelihood estimates (MLE). The data sets 1 and 2 are survival times of a group of patients suffering from head and neck cancer.

Data Set 1: The data set reported by Efron (1988) represent the survival times of a group of patients suffering from Head and Neck cancer disease and treated using radiotherapy (RT)

6.537 10.42 14.48 16.10 22.70 3441.55 4245.28 49.40 53.62 63
64 83 84 91 108 112 129 133 133 139 140 140 146 149 154
157 160 160 165 146 149 154 157 160 160 165 173 176 218 225
241 248 273 277 297 405 417 420 440 523 583 594 1101 1146
1417

Data Set 2: The data set reported by Efron (1988) represent the survival times of a group of patients suffering from Head and Neck cancer disease and treated using a combination of radiotherapy and chemotherapy (RT+CT).

12.20 23.56 23.74 25.87 31.98 37 41.35 47.38 55.46 58.36
63.47 68.46 78.26 74.47 81.43 84 92 94 110 112 119 127 130
133 140 146 155 159 173 179 194 195 209 249 281 319 339
432 469 519 633 725 817 1776

Table 1. MLE's, S.E., -2ln L, AIC, K-S Statistic and p-values of the fitted distributions of data sets 1-2

Data	Model	Parameter Estimate	S.E.	-2ln L	AIC	K-S Statistic	p-value
1	TPWLD	$\hat{\theta} = 0.0046940$ $\hat{\alpha} = 0.05090197$ $\hat{\beta} = -0.0913944$	0.000599 0.009192 0.178881	744.04	**750.04**	**0.172**	**0.064**
	WLD	$\hat{\theta}$ =0.0052993 $\hat{\alpha}$ =0.2124576	0.000797 0.113080	746.79	750.79	0.1826	0.042
	Lindley	$\hat{\theta}$ =0.008804	0.008060	763.74	765.74	0.246	0.002
2	TPWLD	$\hat{\theta}$ =0.0047801 $\hat{\alpha}$ =0.0484017 $\hat{\beta}$ = -0.077115	0.0007015 0.0103219 0.1822874	563.45	**569.45**	**0.146**	**0.281**
	WLD	$\hat{\theta}$ = 0.0054135 $\hat{\alpha}$ =0.2271618	0.0009513 0.1386034	565.93	569.93	0.161	0.185
	Lindley	$\hat{\theta}$ =0.008905	0.0009409	579.16	581.16	0.219	0.024

It is obvious from the goodness of fit in the above table that TPWLD gives much closer fit than the two-parameter WLD and the one-parameter Lindley distribution, and hence it can be considered as an important tool for modelling survival time data over these distributions.

The fitted plots of TPWLD, WLD and Lindley distribution for data set 1 and 2 are shown in the following Figure 4.

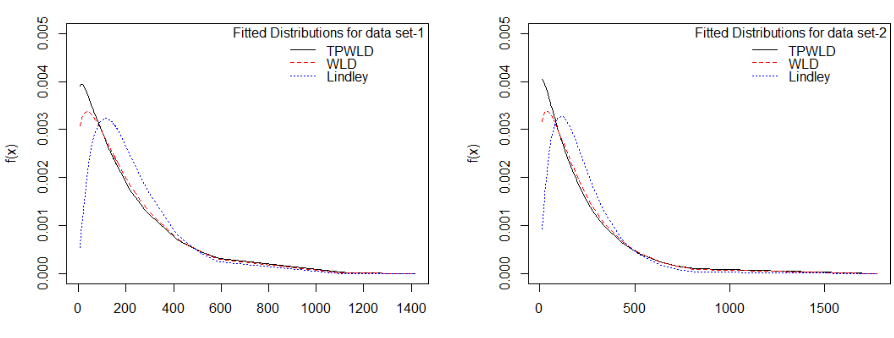


Figure 4. Fitted plots of TPWLD, WLD and Lindley distribution for the data set 1 and 2

8. Concluding remarks

A three-parameter weighted Lindley distribution (TPWLD), which includes two-parameter WLD and Lindley distribution as special cases, has been introduced. Its moments and moments-based expressions, including the coefficient of variation, skewness, kurtosis, and index of dispersion, have been derived and studied. The hazard rate function and the mean residual life function have been obtained and discussed. MLE has been used to estimate the parameters of the distribution. Goodness of fit of TPWLD has been discussed with some survival time data of a group of patients suffering from head and neck cancer and the fit shows a quite satisfactory fit over one-parameter Lindley distribution and two-parameter WLD.

Acknowledgement

Authors are grateful to the editor-in-chief of the journal and anonymous reviewers for their useful comments, which improved the presentation of the paper.

REFERENCES

EFRON, B., (1988). Logistic regression, survival analysis and the Kaplan-Meier curve, Journal of the American Statistical Association, 83, pp. 414–425.

GHITANY, M. E., ATIEH, B., NADARAJAH. S., (2008). Lindley distribution and its Application, Mathematics Computing and Simulation, 78, pp. 493–506.

GHITANY, M. E., ALQALLAF, F., AL-MUTAIRI, D. K., HUSAIN, H. A., (2011). A two-parameter weighted Lindley distribution and its applications to survival data, Mathematics and Computers in simulation, 81, pp. 1190–1201.

LINDLEY, D. V., (1958). Fiducial distributions and Bayes' theorem, Journal of the Royal Statistical Society, Series B, 20, pp. 102–107.

SANKARAN, M., (1970). The discrete Poisson-Lindley distribution, Biometrics, 26(1), pp. 145–149.

SHAKED, M., SHANTHIKUMAR, J. G., (1994). Stochastic Orders and Their Applications, Academic Press, New York.

SHANKER, R., MISHRA, A., (2013 a). A Two Parameter Lindley Distribution, Statistics in Transition new series, 14 (1), pp. 45–56.

SHANKER, R, MISHRA, A., (2013 b). A Quasi Lindley Distribution, African Journal of Mathematics and Computer Science Research (AJMCSR), 6 (4), pp. 64 – 71.

SHANKER, R., SHARMA, S., SHANKER, R., (2013). A Two Parameter Lindley Distribution for Modeling Waiting and Survival Times Data, Applied Mathematics, 4 (2), pp. 363–368.

SHANKER, R., AMANUEL, A. G., (2013). A New Quasi Lindley Distribution**,** International Journal of Statistics and System, 8 (2), pp. 143–156.

SHANKER, R., HAGOS, F., SUJATHA, S., (2015). On modeling of Lifetimes data using exponential and Lindley distributions, Biometrics & Biostatistics International Journal, 2 (5), pp. 1–9.

SHANKER, R., SHUKLA, K. K., HAGOS, F., (2016). On weighted Lindley distribution and Its applications to model Lifetime data, Jacobs Journal of Biostatistics, 1 (1), pp. 1–9.