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IMPROVED ESTIMATION OF THE SCALE PARAMETER FOR LOG-LOGISTIC DISTRIBUTION USING BALANCED RANKED SET SAMPLING

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ABSTRACT

In this article we have suggested some improved estimators of a scale parameter of log-logistic distribution (LLD) under a situation where the units in a sample can be ordered by judgement method without any error. We have also suggested some linear shrinkage estimator of a scale parameter of LLD. Efficiency comparisons are also made in this work.

Key words: minimum mean squared error estimator, shrinkage estimator, log-logistic distribution, best linear unbiased estimator, median ranked set sample.

AMS Subject Classification: 62G30; 62H12.

1. Introduction

Ranked set sampling (RSS) is a method of sampling that can be advantageous when quantification of all sampling units is costly but a small set of units can be easily ranked, according to the character under investigation, without actual quantification. The technique was first introduced by McIntyre (1952) for estimating means pasture and forage yields. The theory and application of ranked set sampling given by Chen *et al.* (2004).

A random variable X is said to have a log-logistic distribution with the scale parameter α and the shape parameter β if its cumulative distribution function (CDF) and probability density function (PDF) are respectively given as (see, Lesitha and Thomas (2012))

$$F(x;\alpha,\beta)=\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}},x>0,\alpha>0,\beta>1 \tag{1}$$

and

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$$f(x;\alpha,\beta)=\frac{\beta\alpha^{\beta}x^{\beta-1}}{\left(\alpha^{\beta}+x^{\beta}\right)^{2}},x>0,\alpha>0,\beta>1. \tag{2}$$

Also, the *kth* moment of (2) exists only when $k<\beta$ and is given by

$$E\left(X^{k}\right)=\alpha^{k}B\left(1-\frac{k}{\beta},1+\frac{k}{\beta}\right), \tag{3}$$

where B denotes beta function.

The applications of log-logistic distribution are well known in a survival analysis of data sets such as survival times of cancer patients in which the hazard rate increases initially and decreases later (for example, see Bennett (1983)). In economic studies of distributions of wealth or income, it is known as Fisk distribution (see Fisk (1961)) and is considered as an equivalent alternative to a lognormal distribution. For further details on the importance and applications of a log-logistic distribution one may refer to Shoukri *et al.* (1988), Geskus (2001), Robson and Reed (1999) and Ahmad *et al.* (1988). For current reference in this context the reader is referred to Singh and Mehta (2013; 2014, a, b, 2015, 2016 a, b, c), Mehta and Singh (2014) and Mehta (2015).

If $X_{1:n},X_{2:n},...,X_{n:n}$ are the order statistics of a random sample of size n drawn from (1) then

$$Y_{r:n}=\frac{X_{r:n}}{\alpha},r=1,2,...,n, \tag{4}$$

are distributed as order statistics of the same sample size drawn from a $LLD(1,\beta)$ with PDF given by

$$g(y,\beta)=\frac{\beta y^{\beta-1}}{\left(1+y^{\beta}\right)^{2}},y>0,\beta>1. \tag{5}$$

For a detailed description of various properties of order statistics arising from a $LLD(1,\beta)$ one may refer to Ragab and Green (1984). Balakrishnan and Malik (1987) have given some recurrence relations on the single and product moments of order statistics arising from a $LLD(1,\beta)$. Suppose

$$\gamma_{r:n}=E\left(Y_{r:n}\right),r=1,2,...,n, \tag{6}$$

$$\sigma_{r,s:n}=Cov\left(Y_{r:n},Y_{s:n}\right),1\le r<s\le n \tag{7}$$

and

$$\sigma_{r,r:n}=Var\left(Y_{r:n}\right),1\le r\le n. \tag{8}$$

By using (4) in (6)-(8) we have

$$E(X_{r:n}) = \alpha\gamma_{r:n}, 1 \le r \le n \tag{9}$$

$$Cov(X_{r:n}, X_{s:n}) = \alpha^2 \sigma_{r,s:n}, 1 \le r < s \le n \tag{10}$$

$$Var(X_{r:n}) = \alpha^2 \sigma_{r,r:n}, 1 \le r \le n. \tag{11}$$

Lesitha and Thomas (2012) have computed the values of $\gamma_{r:n}$ and $\sigma_{r,s:n}, 1 \le r, s \le n$ independently for $n = 2(1)8$ and for $\beta = 2.5(0.5)5.0$ using Mathcad software so as to use those values for the computation of BLUE of α based on order statistics. If $\boldsymbol{X} = (X_{1:n}, X_{2:n}, \ldots, X_{n:n})'$ then the mean vector $E(X)$ and dispersion matrix $D(X)$ of X are

$$E(\boldsymbol{X}) = \gamma\alpha$$

and

$$D(\boldsymbol{X}) = \alpha^2 G,$$

where $\gamma = (\gamma_{1:n}, \gamma_{2:n}, \ldots, \gamma_{n:n})'$ and $\boldsymbol{G} = ((\sigma_{r,s:n}))$.

Thus, by Gauss-Markov theorem Lesitha and Thomas (2012) gives the BLUE $\hat{\alpha}$ based on order statistics of a random sample of size n as:

$$t_1 = \hat{\alpha} = (\gamma' G^{-1}\gamma)^{-1} \gamma' G^{-1} \boldsymbol{X}$$

and

$$Var(t_1) = (\gamma' G^{-1}\gamma)^{-1} \alpha^2 = \alpha^2 V_1, \tag{12}$$

where $V_1 = (\gamma' G^{-1}\gamma)^{-1}$.

Lesitha and Thomas (2012) further estimate α based on the mean of unbiased estimators of α defined from each individual observations in the balanced ranked set sampling as:

$$t_2 = \alpha^* = \frac{1}{n}\sum_{r=1}^{n}\left[\frac{X_{(r:n)r}}{\gamma_{r:n}}\right],$$

with

$$Var(t_2) = \frac{1}{n^2}\sum_{r=1}^{n}\left[\frac{\sigma_{r,r:n}}{\gamma_{r:n}^2}\right]\alpha^2 = \alpha^2 V_2, \tag{13}$$

where $V_2 = \frac{1}{n^2}\sum_{r=1}^{n}\left[\frac{\sigma_{r,r:n}}{\gamma_{r:n}^2}\right]$.

Lesitha and Thomas (2012) also estimate α based on BLUE in the balanced ranked set sampling as:

$$t_3 = \alpha^{**} = \left(\gamma' G_1^{-1} \gamma\right)^{-1} \gamma' G_1^{-1} X_{rss}$$

and

$$Var(t_3) = \left(\gamma' G_1^{-1} \gamma\right)^{-1} \alpha^2 = \alpha^2 V_3 . \tag{14}$$

where $\gamma = \left(\gamma_{1:n}, \gamma_{2:n}, ..., \gamma_{n:n}\right)'$, $G_1 = diag\left(\sigma_{1,1:n}, \sigma_{2,2:n}, ..., \sigma_{n,n:n}\right)$ and $V_3 = \left(\gamma' G_1^{-1} \gamma\right)^{-1}$.

When n is small the estimators α^* and α^{**} may not be acceptable for the expected level of precision. In such situations Lesitha and Thomas (2012) makes N cycles of RSS. For details see Chen *et al.* (2004). Suppose α_i^* and α_i^{**} denote the estimators of α corresponding to α^* and α^{**} respectively, based on the *ith* cycle. Then, estimators of α based on N cycles are given by:

$$\overline{\alpha^*} = \frac{1}{N} \sum_{i=1}^{N} \alpha_i^* ,$$

and

$$\overline{\alpha^{**}} = \frac{1}{N} \sum_{i=1}^{N} \alpha_i^{**}$$

with

$$Var\left(\overline{\alpha^*}\right) = \frac{\alpha^2}{Nn^2} \sum_{r=1}^{N} \left[\frac{\sigma_{r,r:n}}{\gamma_{r:n}^2} \right],$$

and

$$Var\left(\overline{\alpha^{**}}\right) = \frac{\left(\gamma' G_1^{-1} \gamma\right)^{-1} \alpha^2}{N} .$$

Median ranked set sampling (MRSS) was first introduced by Muttlak (1997) to estimate the mean of a normal distribution. In general, MRSS is applied as a modification of RSS when one is interested in estimating a parameter associated with the central tendency of a distribution. The procedures of MRSS are given as: Select n independent samples each with n units as in the case of RSS. Then rank the units in each sample either by judgement method or by using some inexpensive means without having actual measurement on the unit. Lesitha and Thomas (2012) used MRSS method to estimate α as:

$$t_4 = \tilde{\alpha} = \begin{cases} \dfrac{1}{n} \displaystyle\sum_{r=1}^{n} \left[\dfrac{X_{(m:n)r}}{\gamma_{m:n}} \right] & ; \text{ when } m = \dfrac{n+1}{2} \text{ and } n \text{ is odd} \\ \left(\gamma_1' G_2^{-1} \gamma_1\right)^{-1} \gamma_1' G_2^{-1} X_{mrss} & ; \text{when } m = \dfrac{n}{2} \text{ and } n \text{ is even} \end{cases}$$

with

$$Var(t_4)=\begin{cases}\frac{1}{n}\left[\frac{\sigma_{m,m:n}}{\gamma_{m:n}^2}\right]\alpha^2 & ;\ when\ m=\frac{n+1}{2} and\, n\ is\, odd\\ \left(\gamma_1{}' G_2{}^{-1}\gamma_1\right)^{-1}\alpha^2 & ;when\ m=\frac{n}{2} and\, n\ is\, even\end{cases}=\alpha^2 V_4, \tag{15}$$

where

$$X_{mrss}=\left(X_{(m:n)1},X_{(m+1:n)2},X_{(m:n)3},X_{(m+1:n)4},\dots,X_{(m+1:n)n}\right)'$$
$$\gamma_1=\left(\gamma_{m:n},\gamma_{m+1:n},\gamma_{m:n},\gamma_{m+1:n},\dots,\gamma_{m+1:n}\right)'$$
$$G_2=diag\left(\sigma_{m,m:n},\sigma_{m+1,m+1:n},\sigma_{m,m:n},\sigma_{m+1,m+1:n},\dots,\sigma_{m+1,m+1:n}\right)\text{ and}$$
$$V_4=\begin{cases}\frac{1}{n}\left[\frac{\sigma_{m,m:n}}{\gamma_{m:n}^2}\right] & ;\ when\ m=\frac{n+1}{2} and\, n\ is\, odd\\ \left(\gamma_1{}' G_2{}^{-1}\gamma_1\right)^{-1} & ;when\ m=\frac{n}{2} and\, n\ is\, even\end{cases}.$$

2. Improved estimation of the scale parameter α

Let $t_i, i=1,2,3,4$ be an unbiased estimator of the parameter α, then we define a class of estimators for α as

$$T_i=A_i t_i, i=1,2,3,4,$$

where $A_i's, i=1,2,3,4$ are suitably chosen constants such that mean squared error of the estimators $T_i's, i=1,2,3,4$ is minimum.

The biases and mean squared errors (MSEs) of $T_i, i=1,2,3,4$ are respectively given by

$$B(T_i)=\alpha(A_i-1),$$

and

$$MSE(T_i)=A_i^2 Var(t_i)+(A_i-1)^2\alpha^2=\alpha^2\left[A_i^2(1+V_i)-2A_i+1\right].$$

The $MSE(T_i), i=1,2,3,4$ is minimized for

$$A_i=(1+V_i)^{-1}, i=1,2,3,4.$$

Thus, the resulting minimum MSE estimator of α is given by

$$T_{0i}=t_i(1+V_i)^{-1}, i=1,2,3,4.$$

The biases and MSEs of $T_{0i}, i = 1,2,3,4$ are respectively given as

$$B(T_{0i}) = -\alpha\left(\frac{V_i}{1+V_i}\right), \tag{16}$$

and

$$MSE(T_{0i}) = \alpha^2\left(\frac{V_i}{1+V_i}\right). \tag{17}$$

We have from (12) - (15) and (17) that

$$Var(t_i) - MSE(T_{0i}) = \alpha^2 \frac{V_i^2}{(1+V_i)} > 0, i = 1,2,3,4. \tag{18}$$

It follows from (18) that the proposed MMSE estimators $T_{0i}'s, i = 1,2,3,4$ are better than the corresponding usual unbiased estimators $t_i's, i = 1,2,3,4$.

3. Improved estimation of the scale parameter α with prior information

Let $t_i, i = 1,2,3,4$ be an unbiased estimator of the parameter α, then we define a class of estimators of α using the prior point estimate α_0 of α as

$$T_{1i} = \alpha_0 + B_i t_i, i = 1,2,3,4, \tag{19}$$

where $B_i's, i = 1,2,3,4$ are suitably chosen constants such that mean squared error of the estimators $T_{1i}'s, i = 1,2,3,4$ are minimum.

The biases and mean squared errors (MSEs) of $T_{1i}, i = 1,2,3,4$ are respectively given by

$$B(T_{1i}) = \alpha(\phi + B_i),$$

and

$$MSE(T_{1i}) = \alpha^2\left[\phi^2 + B_i^2(1+V_i) + 2\phi B_i\right].$$

where $\phi = \left(\frac{\alpha_0}{\alpha} - 1\right) = (\lambda - 1)$ with $\lambda = \frac{\alpha_0}{\alpha}$.

The $MSE(T_{1i}), i = 1,2,3,4$ is minimized for

$$B_i = -\phi(1+V_i)^{-1}, i = 1,2,3,4. \tag{20}$$

The value of $B_i, i=1,2,3,4$ at (20) depends on the unknown parameter α, so an estimate of $B_i, i=1,2,3,4$ based on sample data is given by

$$B_i^* = -\frac{\phi^*}{(1+V_i)} = -\frac{(\theta_{20}-t_i)}{t_i(1+V_i)}, i=1,2,3,4.$$

Putting $B_i^*, i=1,2,3,4$ in (19), we get a shrinkage estimator of $t_i's, i=1,2,3,4$ as

$$T_{1i}^* = \alpha_0 - (1+V_i)^{-1}(\alpha_0 - t_i), i=1,2,3,4. \quad (21)$$

The biases and mean squared errors (MSEs) of the estimators $T_{1i}^*'s, i=1,2,3,4$ are respectively given by

$$B(T_{1i}^*) = \alpha\phi\left(\frac{V_i}{1+V_i}\right), \quad (22)$$

and

$$MSE(T_{1i}^*) = \alpha^2 \frac{V_i(\phi^2 V_i + 1)}{(1+V_i)^2}. \quad (23)$$

Comparisons of the proposed shrinkage estimators $T_{1i}^*'s, i=1,2,3,4$ with that of corresponding usual unbiased estimators $t_i's, i=1,2,3,4$ are given in the following Theorem 1.

Theorem 1: *The proposed shrinkage estimators* $T_{1i}^*'s, i=1,2,3,4$ *are better than the corresponding usual unbiased estimators* $t_i's, i=1,2,3,4$ *if*

$$\lambda \in \left(0,\left(1+\sqrt{(2+V_i)}\right)\right),$$

i.e. if $\alpha_0 \in \left(0, \alpha\left\{1+\sqrt{(2+V_i)}\right\}\right)$,

i.e. if $\alpha \in \left(\frac{\alpha_0}{\left\{1+\sqrt{(2+V_i)}\right\}}, \infty\right)$.

Proof: *From (12) - (15) and (23), we have that*

$MSE(T_{1i}^*) < Var(t_i), i=1,2,3,4$ *if*

$$\alpha^2 \frac{V_i(\phi^2 V_i + 1)}{(1+V_i)^2} < \alpha^2 V_i,$$

i.e. if $\frac{(\phi^2 V_i + 1)}{(1+V_i)^2} < 1$,

$$i.e.\ if \quad \phi^2 < \frac{\left((1+V_i)^2 - 1\right)}{V_i},$$

$$i.e.\ if \quad \phi^2 < \frac{V_i(2+V_i)}{V_i},$$

$$i.e.\ if \quad \phi^2 < 2+V_i\ ,$$

$$i.e.\ if \ (\lambda-1)^2 < 2+V_i\,,$$

$$i.e.\ if \ \left\{1-\sqrt{(2+V_i)}\right\} < \lambda < \left\{1+\sqrt{(2+V_i)}\right\}. \tag{24}$$

Since $\left\{1-\sqrt{(2+V_i)}\right\} < 0$ *and* $\lambda(=\alpha_0/\alpha)$ *cannot be negative therefore (24) reduces to*

$$0 < \lambda < \left\{1+\sqrt{(2+V_i)}\right\},$$

or

$$0 < \alpha_0 < \alpha\left\{1+\sqrt{(2+V_i)}\right\},$$

or

$$\frac{\alpha_0}{\left\{1+\sqrt{(2+V_i)}\right\}} < \alpha < \infty .$$

Hence the theorem. ♦

Further, we have compared the proposed shrinkage estimators $T_{1i}^{*}{}'s, i=1,2,3,4$ with that of corresponding MMSE estimators $T_{0i}{}'s, i=1,2,3,4$ and the results are presented in Theorem 2.

Theorem 2: *The proposed shrinkage estimators* $T_{1i}^{*}{}'s, i=1,2,3,4$ *are better than the corresponding MMSE estimators* $T_{0i}{}'s, i=1,2,3,4$ *if*

$$\lambda \in (0,2),$$

$$i.e.\ if \ \alpha_0 \in (0,2\alpha),$$

$$i.e.\ if \ \alpha \in \left(\frac{\alpha_0}{2}, \infty\right).$$

Proof: *From (17) and (23) we have that*

$$MSE\left(T_{1i}^{*}\right) < MSE\left(T_{0i}\right), i=1,2,...,7 \ if$$

$$\alpha^2 \frac{V_i\left(\phi^2 V_i + 1\right)}{(1+V_i)^2} < \alpha^2 \frac{V_i}{1+V_i},$$

i.e. if $\frac{(\phi^2 V_i + 1)}{(1+V_i)} < 1$,

i.e. if $\phi^2 < 1$,

i.e. if $-1 < \phi < 1$,

i.e. if $0 < \lambda < 2$,

or $0 < \alpha_0 < 2\alpha$,

or $\frac{\alpha_0}{2} < \alpha < \infty$.

Hence the theorem. ♦

4. Relative efficiencies

We have computed the relative efficiencies of various suggested estimators to usual estimators by using the formulae:

$$e_1 = RE(T_{01}, t_1) = 1 + V_1;$$
$$e_2 = RE(T_{02}, t_2) = 1 + V_2;$$
$$e_3 = RE(T_{03}, t_3) = 1 + V_3;$$
$$e_4 = RE(T_{04}, t_4) = 1 + V_4;$$
$$e_5 = RE(T_{04}, T_{01}) = \frac{V_1(1+V_4)}{V_4(1+V_1)};$$
$$e_6 = RE(T_{04}, T_{02}) = \frac{V_2(1+V_4)}{V_4(1+V_2)};$$
$$e_7 = RE(T_{04}, T_{03}) = \frac{V_3(1+V_4)}{V_4(1+V_3)};$$
$$e_8 = RE(T_{11}^*, t_1) = \frac{(1+V_1)^2}{(\phi^2 V_1 + 1)};$$
$$e_9 = RE(T_{11}^*, T_{01}) = \frac{(1+V_1)}{(\phi^2 V_1 + 1)};$$
$$e_{10} = RE(T_{12}^*, t_2) = \frac{(1+V_2)^2}{(\phi^2 V_2 + 1)};$$
$$e_{11} = RE(T_{12}^*, T_{02}) = \frac{(1+V_2)}{(\phi^2 V_2 + 1)};$$
$$e_{12} = RE(T_{13}^*, t_3) = \frac{(1+V_3)^2}{(\phi^2 V_3 + 1)};$$
$$e_{13} = RE(T_{13}^*, T_{03}) = \frac{(1+V_3)}{(\phi^2 V_3 + 1)};$$
$$e_{14} = RE(T_{14}^*, t_4) = \frac{(1+V_4)^2}{(\phi^2 V_4 + 1)};$$

and

$$e_{15} = RE(T_{14}^*, T_{04}) = \frac{(1+V_4)}{(\phi^2 V_4 + 1)}.$$

- The values of $e_i, i = 1,2,...,7$ are shown in Table 1 for $n = 2(1)8$ and $\beta = 2.5(0.5)5$.
- The values of $e_i, i = 8,9,...,15$ are shown in Tables 2 to 5 for $n = 2(1)8;$ $\beta = 2.5(0.5)5$ and different values of $\lambda = \frac{\alpha_0}{\alpha}$.

Table 1. The values of e_i's, $i = 1,2,...,7$.

n	*β*	e_1	e_2	e_3	e_4	e_5	e_6	e_7
2	2.5	1.3371	1.4025	1.2799	1.2799	1.1530	1.3124	1.0000
	3.0	1.2158	1.1978	1.1674	1.1674	1.2381	1.1516	1.0000
	3.5	1.1514	1.1254	1.1135	1.1135	1.2901	1.0932	1.0000
	4.0	1.1126	1.0885	1.0827	1.0827	1.3242	1.0643	1.0000
	4.5	1.0872	1.0665	1.0633	1.0633	1.3476	1.0474	1.0000
	5.0	1.0697	1.0521	1.0501	1.0501	1.3644	1.0368	1.0000
3	2.5	1.2011	1.1904	1.1180	1.0832	2.1798	2.0828	1.3740
	3.0	1.1308	1.0970	1.0752	1.0543	2.2446	1.7159	1.3572
	3.5	1.0937	1.0626	1.0526	1.0385	2.3093	1.5893	1.3475
	4.0	1.0706	1.0447	1.0391	1.0288	2.3527	1.5272	1.3422
	4.5	1.0552	1.0338	1.0303	1.0224	2.3815	1.4910	1.3382
	5.0	1.0443	1.0266	1.0242	1.0180	2.4033	1.4675	1.3356
4	2.5	1.1392	1.1118	1.0648	1.0477	2.6846	2.2089	1.3376
	3.0	1.0939	1.0582	1.0424	1.0317	2.7929	1.7903	1.3246
	3.5	1.0678	1.0380	1.0301	1.0227	2.8598	1.6495	1.3176
	4.0	1.0514	1.0273	1.0226	1.0171	2.9034	1.5803	1.3126
	4.5	1.0413	1.0208	1.0176	1.0134	3.0062	1.5408	1.3102
	5.0	1.0324	1.0164	1.0141	1.0107	2.9570	1.5158	1.3085
5	2.5	1.1078	1.0740	1.0409	1.0278	3.5999	2.5482	1.4547
	3.0	1.0733	1.0391	1.0272	1.0187	3.7126	2.0476	1.4397
	3.5	1.0531	1.0257	1.0195	1.0135	3.7819	1.8805	1.4317
	4.0	1.0403	1.0186	1.0147	1.0101	3.8704	1.8188	1.4421
	4.5	1.0316	1.0141	1.0115	1.0080	3.8489	1.7503	1.4223
	5.0	1.0256	1.0112	1.0092	1.0065	3.8785	1.7199	1.4196
6	2.5	1.0880	1.0528	1.0282	1.0196	4.2168	2.6150	1.4288
	3.0	1.0600	1.0282	1.0189	1.0133	4.3313	2.0988	1.4170
	3.5	1.0437	1.0187	1.0136	1.0096	4.3995	1.9265	1.4101
	4.0	1.0332	1.0135	1.0103	1.0073	4.4473	1.8430	1.4065
	4.5	1.0261	1.0103	1.0080	1.0057	4.4756	1.7943	1.4024
	5.0	1.0211	1.0082	1.0065	1.0046	4.5010	1.7638	1.4009
7	2.5	1.0735	1.0397	1.0206	1.0138	5.0286	2.8038	1.4800
	3.0	1.0509	1.0214	1.0139	1.0094	5.1991	2.2498	1.4690
	3.5	1.0372	1.0147	1.0106	1.0068	5.2926	2.1298	1.5419
	4.0	1.0282	1.0103	1.0076	1.0052	5.3054	1.9708	1.4542
	4.5	1.0222	1.0079	1.0060	1.0041	5.3447	1.9217	1.4556
	5.0	1.0179	1.0062	1.0048	1.0033	5.3582	1.8854	1.4494
8	2.5	1.0643	1.0310	1.0157	1.0107	5.7315	2.8525	1.4632
	3.0	1.0444	1.0168	1.0106	1.0073	5.8758	2.2888	1.4526
	3.5	1.0329	1.0112	1.0077	1.0053	6.0530	2.1067	1.4484
	4.0	1.0245	1.0081	1.0058	1.0040	5.9651	2.0116	1.4441
	4.5	1.0190	1.0062	1.0046	1.0032	5.9067	1.9593	1.4396
	5.0	1.0156	1.0049	1.0037	1.0025	6.0905	1.9479	1.4568

Table 2. The values of e_i for $i = 8 \; and \; 9$

n	*β*	$e_8 = RE(T^*_{11}, t_1)$						Range of *λ* in which T^*_{11} is efficient to t_1
		λ = ***1.00***	*λ* = ***1.25*** and *λ* = ***0.75***	*λ* = ***1.50*** and *λ* = ***0.50***	*λ* = ***1.75*** and *λ* = ***0.25***	*λ* = ***2.00***	*λ* = ***2.25***	
2	2.5	1.7878	1.7509	1.6489	1.5028	1.3371	1.1710	(0,2.53)
	3.0	1.4782	1.4586	1.4026	1.3182	1.2158	1.1054	(0,2.49)
	3.5	1.3257	1.3133	1.2773	1.2217	1.1514	1.0721	(0,2.47)
	4.0	1.2378	1.2292	1.2039	1.1641	1.1126	1.0527	(0,2.45)
	4.5	1.1820	1.1756	1.1568	1.1268	1.0872	1.0403	(0,2.44)
	5.0	1.1442	1.1392	1.1246	1.1010	1.0697	1.0319	(0,2.44)
3	2.5	1.4426	1.4247	1.3735	1.2960	1.2011	1.0977	(0,2.48)
	3.0	1.2786	1.2683	1.2382	1.1910	1.1308	1.0617	(0,2.46)
	3.5	1.1961	1.1892	1.1688	1.1363	1.0937	1.0434	(0,2.45)
	4.0	1.1461	1.1411	1.1263	1.1024	1.0706	1.0323	(0,2.44)
	4.5	1.1133	1.1095	1.0982	1.0798	1.0552	1.0250	(0,2.43)
	5.0	1.0906	1.0876	1.0787	1.0641	1.0443	1.0200	(0,2.43)
4	2.5	1.2977	1.2865	1.2541	1.2035	1.1392	1.0659	(0,2.46)
	3.0	1.1966	1.1896	1.1692	1.1366	1.0939	1.0435	(0,2.45)
	3.5	1.1402	1.1354	1.1212	1.0983	1.0678	1.0310	(0,2.44)
	4.0	1.1053	1.1018	1.0913	1.0743	1.0514	1.0232	(0,2.43)
	4.5	1.0843	1.0815	1.0732	1.0597	1.0413	1.0186	(0,2.43)
	5.0	1.0659	1.0638	1.0574	1.0468	1.0324	1.0145	(0,2.43)
5	2.5	1.2272	1.2190	1.1950	1.1570	1.1078	1.0503	(0,2.45)
	3.0	1.1519	1.1466	1.1312	1.1063	1.0733	1.0336	(0,2.44)
	3.5	1.1091	1.1054	1.0945	1.0769	1.0531	1.0241	(0,2.43)
	4.0	1.0823	1.0796	1.0715	1.0583	1.0403	1.0181	(0,2.43)
	4.5	1.0643	1.0622	1.0559	1.0457	1.0316	1.0141	(0,2.43)
	5.0	1.0518	1.0501	1.0451	1.0369	1.0256	1.0114	(0,2.42)
6	2.5	1.1837	1.1772	1.1582	1.1279	1.0880	1.0406	(0,2.44)
	3.0	1.1237	1.1195	1.1071	1.0870	1.0600	1.0273	(0,2.44)
	3.5	1.0892	1.0863	1.0775	1.0631	1.0437	1.0197	(0,2.43)
	4.0	1.0675	1.0653	1.0587	1.0479	1.0332	1.0149	(0,2.43)
	4.5	1.0529	1.0512	1.0461	1.0377	1.0261	1.0116	(0,2.42)
	5.0	1.0426	1.0413	1.0372	1.0304	1.0211	1.0094	(0,2.42)
7	2.5	1.1524	1.1471	1.1316	1.1066	1.0735	1.0337	(0,2.44)
	3.0	1.1043	1.1008	1.0905	1.0736	1.0509	1.0230	(0,2.43)
	3.5	1.0759	1.0734	1.0659	1.0538	1.0372	1.0167	(0,2.43)
	4.0	1.0572	1.0554	1.0498	1.0407	1.0282	1.0126	(0,2.42)
	4.5	1.0449	1.0434	1.0391	1.0320	1.0222	1.0099	(0,2.42)
	5.0	1.0362	1.0350	1.0316	1.0258	1.0179	1.0079	(0,2.42)
8	2.5	1.1327	1.1282	1.1148	1.0932	1.0643	1.0293	(0,2.44)
	3.0	1.0907	1.0877	1.0787	1.0641	1.0444	1.0200	(0,2.43)
	3.5	1.0669	1.0647	1.0582	1.0475	1.0329	1.0147	(0,2.43)
	4.0	1.0497	1.0481	1.0433	1.0354	1.0245	1.0109	(0,2.42)
	4.5	1.0384	1.0372	1.0335	1.0274	1.0190	1.0084	(0,2.42)
	5.0	1.0315	1.0305	1.0275	1.0225	1.0156	1.0069	(0,2.42)

Table 2. The values of e_i for $i = 8$ *and* 9 (cont.)

n	β	$e_9 = RE\left(T_{11}^*, T_{01}\right)$ $\lambda = 1.00$	$\lambda = 1.25$ and $\lambda = 0.75$	$\lambda = 1.50$ and $\lambda = 0.50$	$\lambda = 1.75$ and $\lambda = 0.25$	$\lambda = 2.00$ and $\lambda = 0.00$	Range of λ in which T_{11}^* is efficient to T_{01}
2	2.5	1.3371	1.3095	1.2332	1.1240	1.0000	[0,2]
	3.0	1.2158	1.1996	1.1536	1.0842	1.0000	[0,2]
	3.5	1.1514	1.1406	1.1094	1.0610	1.0000	[0,2]
	4.0	1.1126	1.1048	1.0821	1.0463	1.0000	[0,2]
	4.5	1.0872	1.0813	1.0640	1.0364	1.0000	[0,2]
	5.0	1.0697	1.0650	1.0514	1.0293	1.0000	[0,2]
3	2.5	1.2011	1.1862	1.1436	1.0790	1.0000	[0,2]
	3.0	1.1308	1.1216	1.0950	1.0533	1.0000	[0,2]
	3.5	1.0937	1.0873	1.0687	1.0389	1.0000	[0,2]
	4.0	1.0706	1.0659	1.0520	1.0297	1.0000	[0,2]
	4.5	1.0552	1.0515	1.0408	1.0234	1.0000	[0,2]
	5.0	1.0443	1.0414	1.0329	1.0189	1.0000	[0,2]
4	2.5	1.1392	1.1294	1.1009	1.0565	1.0000	[0,2]
	3.0	1.0939	1.0875	1.0688	1.0390	1.0000	[0,2]
	3.5	1.0678	1.0633	1.0500	1.0286	1.0000	[0,2]
	4.0	1.0514	1.0480	1.0380	1.0218	1.0000	[0,2]
	4.5	1.0413	1.0386	1.0307	1.0177	1.0000	[0,2]
	5.0	1.0324	1.0304	1.0241	1.0139	1.0000	[0,2]
5	2.5	1.1078	1.1004	1.0787	1.0445	1.0000	[0,2]
	3.0	1.0733	1.0684	1.0540	1.0308	1.0000	[0,2]
	3.5	1.0531	1.0496	1.0393	1.0226	1.0000	[0,2]
	4.0	1.0403	1.0377	1.0300	1.0173	1.0000	[0,2]
	4.5	1.0316	1.0296	1.0235	1.0136	1.0000	[0,2]
	5.0	1.0256	1.0239	1.0191	1.0110	1.0000	[0,2]
6	2.5	1.0880	1.0820	1.0646	1.0367	1.0000	[0,2]
	3.0	1.0600	1.0561	1.0444	1.0254	1.0000	[0,2]
	3.5	1.0437	1.0408	1.0324	1.0186	1.0000	[0,2]
	4.0	1.0332	1.0311	1.0247	1.0143	1.0000	[0,2]
	4.5	1.0261	1.0244	1.0195	1.0113	1.0000	[0,2]
	5.0	1.0211	1.0197	1.0157	1.0091	1.0000	[0,2]
7	2.5	1.0735	1.0686	1.0541	1.0309	1.0000	[0,2]
	3.0	1.0509	1.0475	1.0377	1.0216	1.0000	[0,2]
	3.5	1.0372	1.0348	1.0277	1.0160	1.0000	[0,2]
	4.0	1.0282	1.0264	1.0210	1.0122	1.0000	[0,2]
	4.5	1.0222	1.0208	1.0166	1.0096	1.0000	[0,2]
	5.0	1.0179	1.0168	1.0134	1.0078	1.0000	[0,2]
8	2.5	1.0643	1.0600	1.0474	1.0271	1.0000	[0,2]
	3.0	1.0444	1.0415	1.0329	1.0189	1.0000	[0,2]
	3.5	1.0329	1.0308	1.0245	1.0141	1.0000	[0,2]
	4.0	1.0245	1.0230	1.0183	1.0106	1.0000	[0,2]
	4.5	1.0190	1.0178	1.0142	1.0082	1.0000	[0,2]
	5.0	1.0156	1.0146	1.0117	1.0068	1.0000	[0,2]

Table 3. The values of e_i for $i = 10\ and\ 11$

n	β	$e_{10} = RE\left(T_{12}^{*}, t_2\right)$						Range of λ in which T_{12}^{*} is efficient to t_2
		$\lambda = 1.00$	$\lambda = 1.25$ and $\lambda = 0.75$	$\lambda = 1.50$ and $\lambda = 0.50$	$\lambda = 1.75$ and $\lambda = 0.25$	$\lambda = 2.00$	$\lambda = 2.25$	
2	2.5	1.9669	1.9186	1.7871	1.6038	1.4025	1.2075	(0,2.55)
	3.0	1.4347	1.4171	1.3671	1.2910	1.1978	1.0960	(0,2.48)
	3.5	1.2665	1.2566	1.2280	1.1830	1.1254	1.0590	(0,2.46)
	4.0	1.1849	1.1784	1.1592	1.1287	1.0885	1.0409	(0,2.45)
	4.5	1.1374	1.1327	1.1188	1.0964	1.0665	1.0304	(0,2.44)
	5.0	1.1069	1.1033	1.0926	1.0754	1.0521	1.0236	(0,2.43)
3	2.5	1.4171	1.4004	1.3527	1.2800	1.1904	1.0922	(0,2.48)
	3.0	1.2034	1.1961	1.1749	1.1411	1.0970	1.0450	(0,2.45)
	3.5	1.1292	1.1248	1.1118	1.0908	1.0626	1.0285	(0,2.44)
	4.0	1.0914	1.0884	1.0794	1.0646	1.0447	1.0202	(0,2.43)
	4.5	1.0688	1.0665	1.0598	1.0488	1.0338	1.0151	(0,2.43)
	5.0	1.0539	1.0522	1.0470	1.0384	1.0266	1.0119	(0,2.42)
4	2.5	1.2360	1.2274	1.2024	1.1629	1.1118	1.0523	(0,2.45)
	3.0	1.1199	1.1158	1.1038	1.0843	1.0582	1.0265	(0,2.43)
	3.5	1.0775	1.0749	1.0673	1.0549	1.0380	1.0171	(0,2.43)
	4.0	1.0554	1.0536	1.0482	1.0394	1.0273	1.0122	(0,2.42)
	4.5	1.0419	1.0406	1.0366	1.0299	1.0208	1.0092	(0,2.42)
	5.0	1.0330	1.0320	1.0288	1.0236	1.0164	1.0072	(0,2.42)
5	2.5	1.1534	1.1481	1.1325	1.1073	1.0740	1.0339	(0,2.44)
	3.0	1.0798	1.0771	1.0693	1.0565	1.0391	1.0176	(0,2.43)
	3.5	1.0521	1.0504	1.0454	1.0371	1.0257	1.0115	(0,2.42)
	4.0	1.0375	1.0363	1.0327	1.0267	1.0186	1.0082	(0,2.42)
	4.5	1.0285	1.0276	1.0249	1.0204	1.0141	1.0062	(0,2.42)
	5.0	1.0225	1.0218	1.0196	1.0161	1.0112	1.0049	(0,2.42)
6	2.5	1.1084	1.1047	1.0939	1.0764	1.0528	1.0239	(0,2.43)
	3.0	1.0572	1.0554	1.0498	1.0407	1.0282	1.0126	(0,2.42)
	3.5	1.0377	1.0365	1.0328	1.0269	1.0187	1.0083	(0,2.42)
	4.0	1.0272	1.0263	1.0237	1.0194	1.0135	1.0060	(0,2.42)
	4.5	1.0207	1.0201	1.0181	1.0148	1.0103	1.0045	(0,2.42)
	5.0	1.0164	1.0159	1.0143	1.0117	1.0082	1.0036	(0,2.42)
7	2.5	1.0809	1.0783	1.0703	1.0573	1.0397	1.0178	(0,2.43)
	3.0	1.0433	1.0419	1.0377	1.0308	1.0214	1.0095	(0,2.42)
	3.5	1.0295	1.0286	1.0258	1.0211	1.0147	1.0065	(0,2.42)
	4.0	1.0207	1.0200	1.0181	1.0148	1.0103	1.0045	(0,2.42)
	4.5	1.0158	1.0153	1.0138	1.0113	1.0079	1.0035	(0,2.42)
	5.0	1.0125	1.0121	1.0109	1.0090	1.0062	1.0027	(0,2.42)
8	2.5	1.0629	1.0609	1.0548	1.0447	1.0310	1.0138	(0,2.43)
	3.0	1.0339	1.0328	1.0296	1.0242	1.0168	1.0074	(0,2.42)
	3.5	1.0225	1.0218	1.0197	1.0161	1.0112	1.0049	(0,2.42)
	4.0	1.0163	1.0158	1.0143	1.0117	1.0081	1.0036	(0,2.42)
	4.5	1.0125	1.0121	1.0109	1.0090	1.0062	1.0027	(0,2.42)
	5.0	1.0099	1.0096	1.0087	1.0071	1.0049	1.0022	(0,2.42)

Table 3. The values of e_i for $i = 10$ *and* 11 (cont.)

n	β	$e_{11} = RE\left(T_{12}^{*}, T_{02}\right)$					Range of λ in which T_{12}^{*} is efficient to T_{02}
		$\lambda = 1.00$	$\lambda = 1.25$ and $\lambda = 0.75$	$\lambda = 1.50$ and $\lambda = 0.50$	$\lambda = 1.75$ and $\lambda = 0.25$	$\lambda = 2.00$ and $\lambda = 0.00$	
2	2.5	1.4025	1.3680	1.2743	1.1436	1.0000	[0,2]
	3.0	1.1978	1.1831	1.1413	1.0779	1.0000	[0,2]
	3.5	1.1254	1.1166	1.0912	1.0512	1.0000	[0,2]
	4.0	1.0885	1.0825	1.0650	1.0369	1.0000	[0,2]
	4.5	1.0665	1.0621	1.0491	1.0280	1.0000	[0,2]
	5.0	1.0521	1.0487	1.0386	1.0221	1.0000	[0,2]
3	2.5	1.1904	1.1764	1.1363	1.0752	1.0000	[0,2]
	3.0	1.0970	1.0904	1.0710	1.0402	1.0000	[0,2]
	3.5	1.0626	1.0585	1.0463	1.0265	1.0000	[0,2]
	4.0	1.0447	1.0418	1.0332	1.0191	1.0000	[0,2]
	4.5	1.0338	1.0316	1.0252	1.0145	1.0000	[0,2]
	5.0	1.0266	1.0249	1.0198	1.0115	1.0000	[0,2]
4	2.5	1.1118	1.1040	1.0815	1.0460	1.0000	[0,2]
	3.0	1.0582	1.0544	1.0430	1.0247	1.0000	[0,2]
	3.5	1.0380	1.0356	1.0282	1.0163	1.0000	[0,2]
	4.0	1.0273	1.0256	1.0203	1.0118	1.0000	[0,2]
	4.5	1.0208	1.0194	1.0155	1.0090	1.0000	[0,2]
	5.0	1.0164	1.0153	1.0122	1.0071	1.0000	[0,2]
5	2.5	1.0740	1.0690	1.0545	1.0311	1.0000	[0,2]
	3.0	1.0391	1.0366	1.0291	1.0167	1.0000	[0,2]
	3.5	1.0257	1.0241	1.0192	1.0111	1.0000	[0,2]
	4.0	1.0186	1.0174	1.0139	1.0080	1.0000	[0,2]
	4.5	1.0141	1.0132	1.0106	1.0061	1.0000	[0,2]
	5.0	1.0112	1.0105	1.0084	1.0049	1.0000	[0,2]
6	2.5	1.0528	1.0493	1.0391	1.0224	1.0000	[0,2]
	3.0	1.0282	1.0264	1.0210	1.0122	1.0000	[0,2]
	3.5	1.0187	1.0175	1.0139	1.0081	1.0000	[0,2]
	4.0	1.0135	1.0126	1.0101	1.0059	1.0000	[0,2]
	4.5	1.0103	1.0097	1.0077	1.0045	1.0000	[0,2]
	5.0	1.0082	1.0076	1.0061	1.0036	1.0000	[0,2]
7	2.5	1.0397	1.0371	1.0295	1.0170	1.0000	[0,2]
	3.0	1.0214	1.0200	1.0160	1.0093	1.0000	[0,2]
	3.5	1.0147	1.0137	1.0110	1.0064	1.0000	[0,2]
	4.0	1.0103	1.0097	1.0077	1.0045	1.0000	[0,2]
	4.5	1.0079	1.0074	1.0059	1.0034	1.0000	[0,2]
	5.0	1.0062	1.0058	1.0047	1.0027	1.0000	[0,2]
8	2.5	1.0310	1.0290	1.0231	1.0133	1.0000	[0,2]
	3.0	1.0168	1.0158	1.0126	1.0073	1.0000	[0,2]
	3.5	1.0112	1.0105	1.0084	1.0049	1.0000	[0,2]
	4.0	1.0081	1.0076	1.0061	1.0035	1.0000	[0,2]
	4.5	1.0062	1.0058	1.0047	1.0027	1.0000	[0,2]
	5.0	1.0049	1.0046	1.0037	1.0022	1.0000	[0,2]

Table 4. The values of e_i for $i = 12 \text{ and } 13$

n	β	$e_{12} = RE\left(T^*_{13}, t_3\right)$						Range of λ in which T^*_{13} is efficient to t_3
		$\lambda = 1.00$	$\lambda = 1.25$ and $\lambda = 0.75$	$\lambda = 1.50$ and $\lambda = 0.50$	$\lambda = 1.75$ and $\lambda = 0.25$	$\lambda = 2.00$	$\lambda = 2.25$	
2	2.5	1.6380	1.6099	1.5309	1.4152	1.2799	1.1397	(0,2.51)
	3.0	1.3628	1.3487	1.3080	1.2455	1.1674	1.0803	(0,2.47)
	3.5	1.2398	1.2311	1.2056	1.1654	1.1135	1.0531	(0,2.45)
	4.0	1.1723	1.1663	1.1485	1.1202	1.0827	1.0381	(0,2.44)
	4.5	1.1306	1.1262	1.1130	1.0917	1.0633	1.0288	(0,2.44)
	5.0	1.1028	1.0993	1.0891	1.0725	1.0501	1.0227	(0,2.43)
3	2.5	1.2499	1.2407	1.2141	1.1721	1.1180	1.0553	(0,2.46)
	3.0	1.1560	1.1506	1.1347	1.1091	1.0752	1.0345	(0,2.44)
	3.5	1.1080	1.1044	1.0936	1.0761	1.0526	1.0238	(0,2.43)
	4.0	1.0797	1.0771	1.0692	1.0565	1.0391	1.0176	(0,2.43)
	4.5	1.0614	1.0594	1.0535	1.0437	1.0303	1.0135	(0,2.42)
	5.0	1.0489	1.0473	1.0426	1.0348	1.0242	1.0107	(0,2.42)
4	2.5	1.1338	1.1293	1.1158	1.0940	1.0648	1.0296	(0,2.44)
	3.0	1.0867	1.0838	1.0753	1.0613	1.0424	1.0191	(0,2.43)
	3.5	1.0612	1.0592	1.0533	1.0435	1.0301	1.0135	(0,2.42)
	4.0	1.0457	1.0442	1.0398	1.0326	1.0226	1.0100	(0,2.42)
	4.5	1.0355	1.0344	1.0310	1.0253	1.0176	1.0078	(0,2.42)
	5.0	1.0284	1.0275	1.0248	1.0203	1.0141	1.0062	(0,2.42)
5	2.5	1.0835	1.0808	1.0726	1.0592	1.0409	1.0184	(0,2.43)
	3.0	1.0551	1.0533	1.0480	1.0392	1.0272	1.0121	(0,2.42)
	3.5	1.0393	1.0381	1.0343	1.0281	1.0195	1.0086	(0,2.42)
	4.0	1.0295	1.0286	1.0258	1.0211	1.0147	1.0065	(0,2.42)
	4.5	1.0231	1.0223	1.0201	1.0165	1.0115	1.0051	(0,2.42)
	5.0	1.0185	1.0179	1.0162	1.0133	1.0092	1.0041	(0,2.42)
6	2.5	1.0571	1.0553	1.0497	1.0406	1.0282	1.0126	(0,2.42)
	3.0	1.0381	1.0369	1.0332	1.0272	1.0189	1.0084	(0,2.42)
	3.5	1.0274	1.0265	1.0239	1.0196	1.0136	1.0060	(0,2.42)
	4.0	1.0206	1.0200	1.0180	1.0148	1.0103	1.0045	(0,2.42)
	4.5	1.0161	1.0156	1.0141	1.0116	1.0080	1.0035	(0,2.42)
	5.0	1.0130	1.0126	1.0113	1.0093	1.0065	1.0028	(0,2.42)
7	2.5	1.0415	1.0402	1.0362	1.0296	1.0206	1.0091	(0,2.42)
	3.0	1.0279	1.0270	1.0244	1.0200	1.0139	1.0061	(0,2.42)
	3.5	1.0213	1.0206	1.0186	1.0152	1.0106	1.0047	(0,2.42)
	4.0	1.0152	1.0147	1.0133	1.0109	1.0076	1.0033	(0,2.42)
	4.5	1.0119	1.0116	1.0104	1.0086	1.0060	1.0026	(0,2.42)
	5.0	1.0096	1.0093	1.0084	1.0069	1.0048	1.0021	(0,2.42)
8	2.5	1.0316	1.0306	1.0275	1.0226	1.0157	1.0069	(0,2.42)
	3.0	1.0213	1.0207	1.0186	1.0153	1.0106	1.0047	(0,2.42)
	3.5	1.0154	1.0149	1.0135	1.0111	1.0077	1.0034	(0,2.42)
	4.0	1.0117	1.0113	1.0102	1.0084	1.0058	1.0026	(0,2.42)
	4.5	1.0092	1.0089	1.0080	1.0066	1.0046	1.0020	(0,2.42)
	5.0	1.0074	1.0072	1.0065	1.0053	1.0037	1.0016	(0,2.42)

Table 4. The values of e_i for $i = 12\ and\ 13$ (cont.)

n	β	$e_{13} = RE\left(T^*_{13}, T_{03}\right)$					Range of λ in which T^*_{13} is efficient to T_{03}
		$\lambda = 1.00$	$\lambda = 1.25$ and $\lambda = 0.75$	$\lambda = 1.50$ and $\lambda = 0.50$	$\lambda = 1.75$ and $\lambda = 0.25$	$\lambda = 2.00$ and $\lambda = 0.00$	
2	2.5	1.2799	1.2578	1.1962	1.1058	1.0000	[0,2]
	3.0	1.1674	1.1553	1.1205	1.0669	1.0000	[0,2]
	3.5	1.1135	1.1056	1.0828	1.0467	1.0000	[0,2]
	4.0	1.0827	1.0772	1.0608	1.0346	1.0000	[0,2]
	4.5	1.0633	1.0591	1.0467	1.0267	1.0000	[0,2]
	5.0	1.0501	1.0469	1.0371	1.0213	1.0000	[0,2]
3	2.5	1.1180	1.1098	1.0859	1.0484	1.0000	[0,2]
	3.0	1.0752	1.0702	1.0553	1.0316	1.0000	[0,2]
	3.5	1.0526	1.0492	1.0389	1.0224	1.0000	[0,2]
	4.0	1.0391	1.0365	1.0290	1.0167	1.0000	[0,2]
	4.5	1.0303	1.0283	1.0225	1.0130	1.0000	[0,2]
	5.0	1.0242	1.0226	1.0180	1.0104	1.0000	[0,2]
4	2.5	1.0648	1.0605	1.0478	1.0274	1.0000	[0,2]
	3.0	1.0424	1.0397	1.0315	1.0181	1.0000	[0,2]
	3.5	1.0301	1.0282	1.0224	1.0130	1.0000	[0,2]
	4.0	1.0226	1.0211	1.0168	1.0098	1.0000	[0,2]
	4.5	1.0176	1.0165	1.0131	1.0076	1.0000	[0,2]
	5.0	1.0141	1.0132	1.0105	1.0061	1.0000	[0,2]
5	2.5	1.0409	1.0383	1.0304	1.0175	1.0000	[0,2]
	3.0	1.0272	1.0254	1.0203	1.0117	1.0000	[0,2]
	3.5	1.0195	1.0182	1.0145	1.0084	1.0000	[0,2]
	4.0	1.0147	1.0137	1.0110	1.0064	1.0000	[0,2]
	4.5	1.0115	1.0107	1.0086	1.0050	1.0000	[0,2]
	5.0	1.0092	1.0086	1.0069	1.0040	1.0000	[0,2]
6	2.5	1.0282	1.0264	1.0210	1.0121	1.0000	[0,2]
	3.0	1.0189	1.0177	1.0141	1.0082	1.0000	[0,2]
	3.5	1.0136	1.0127	1.0102	1.0059	1.0000	[0,2]
	4.0	1.0103	1.0096	1.0077	1.0045	1.0000	[0,2]
	4.5	1.0080	1.0075	1.0060	1.0035	1.0000	[0,2]
	5.0	1.0065	1.0061	1.0048	1.0028	1.0000	[0,2]
7	2.5	1.0206	1.0193	1.0153	1.0089	1.0000	[0,2]
	3.0	1.0139	1.0130	1.0104	1.0060	1.0000	[0,2]
	3.5	1.0106	1.0099	1.0079	1.0046	1.0000	[0,2]
	4.0	1.0076	1.0071	1.0057	1.0033	1.0000	[0,2]
	4.5	1.0060	1.0056	1.0045	1.0026	1.0000	[0,2]
	5.0	1.0048	1.0045	1.0036	1.0021	1.0000	[0,2]
8	2.5	1.0157	1.0147	1.0117	1.0068	1.0000	[0,2]
	3.0	1.0106	1.0099	1.0079	1.0046	1.0000	[0,2]
	3.5	1.0077	1.0072	1.0057	1.0033	1.0000	[0,2]
	4.0	1.0058	1.0055	1.0044	1.0025	1.0000	[0,2]
	4.5	1.0046	1.0043	1.0034	1.0020	1.0000	[0,2]
	5.0	1.0037	1.0035	1.0028	1.0016	1.0000	[0,2]

Table 5. The values of e_i for $i = 14$ *and* 15

n	β	$e_{14} = RE\left(T_{14}^{*}, t_4\right)$						Range of λ in which T_{14}^{*} is efficient to t_4
		$\lambda = 1.00$	$\lambda = 1.25$ and $\lambda = 0.75$	$\lambda = 1.50$ and $\lambda = 0.50$	$\lambda = 1.75$ and $\lambda = 0.25$	$\lambda = 2.00$	$\lambda = 2.25$	
2	2.5	1.6380	1.6099	1.5309	1.4152	1.2799	1.1397	(0,2.51)
	3.0	1.3628	1.3487	1.3080	1.2455	1.1674	1.0803	(0,2.47)
	3.5	1.2398	1.2311	1.2056	1.1654	1.1135	1.0531	(0,2.45)
	4.0	1.1723	1.1663	1.1485	1.1202	1.0827	1.0381	(0,2.44)
	4.5	1.1306	1.1262	1.1130	1.0917	1.0633	1.0288	(0,2.44)
	5.0	1.1028	1.0993	1.0891	1.0725	1.0501	1.0227	(0,2.43)
3	2.5	1.1733	1.1672	1.1494	1.1209	1.0832	1.0383	(0,2.44)
	3.0	1.1116	1.1078	1.0967	1.0786	1.0543	1.0246	(0,2.43)
	3.5	1.0785	1.0759	1.0682	1.0557	1.0385	1.0173	(0,2.43)
	4.0	1.0585	1.0566	1.0509	1.0416	1.0288	1.0129	(0,2.42)
	4.5	1.0454	1.0439	1.0396	1.0324	1.0224	1.0100	(0,2.42)
	5.0	1.0363	1.0351	1.0316	1.0259	1.0180	1.0080	(0,2.42)
4	2.5	1.0976	1.0944	1.0847	1.0690	1.0477	1.0215	(0,2.43)
	3.0	1.0644	1.0623	1.0561	1.0458	1.0317	1.0142	(0,2.43)
	3.5	1.0459	1.0445	1.0400	1.0327	1.0227	1.0101	(0,2.42)
	4.0	1.0345	1.0334	1.0301	1.0247	1.0171	1.0076	(0,2.42)
	4.5	1.0269	1.0261	1.0235	1.0193	1.0134	1.0059	(0,2.42)
	5.0	1.0216	1.0209	1.0189	1.0155	1.0107	1.0047	(0,2.42)
5	2.5	1.0563	1.0545	1.0490	1.0401	1.0278	1.0124	(0,2.42)
	3.0	1.0378	1.0366	1.0330	1.0270	1.0187	1.0083	(0,2.42)
	3.5	1.0272	1.0264	1.0238	1.0195	1.0135	1.0060	(0,2.42)
	4.0	1.0203	1.0197	1.0178	1.0146	1.0101	1.0045	(0,2.42)
	4.5	1.0161	1.0156	1.0141	1.0116	1.0080	1.0035	(0,2.42)
	5.0	1.0130	1.0126	1.0113	1.0093	1.0065	1.0028	(0,2.42)
6	2.5	1.0395	1.0382	1.0344	1.0282	1.0196	1.0087	(0,2.42)
	3.0	1.0267	1.0258	1.0233	1.0191	1.0133	1.0059	(0,2.42)
	3.5	1.0193	1.0187	1.0169	1.0138	1.0096	1.0042	(0,2.42)
	4.0	1.0146	1.0142	1.0128	1.0105	1.0073	1.0032	(0,2.42)
	4.5	1.0115	1.0111	1.0100	1.0082	1.0057	1.0025	(0,2.42)
	5.0	1.0092	1.0090	1.0081	1.0066	1.0046	1.0020	(0,2.42)
7	2.5	1.0278	1.0269	1.0243	1.0199	1.0138	1.0061	(0,2.42)
	3.0	1.0189	1.0183	1.0165	1.0135	1.0094	1.0041	(0,2.42)
	3.5	1.0137	1.0133	1.0120	1.0098	1.0068	1.0030	(0,2.42)
	4.0	1.0104	1.0101	1.0091	1.0075	1.0052	1.0023	(0,2.42)
	4.5	1.0082	1.0079	1.0071	1.0059	1.0041	1.0018	(0,2.42)
	5.0	1.0066	1.0064	1.0058	1.0047	1.0033	1.0014	(0,2.42)
8	2.5	1.0214	1.0207	1.0187	1.0153	1.0107	1.0047	(0,2.42)
	3.0	1.0146	1.0142	1.0128	1.0105	1.0073	1.0032	(0,2.42)
	3.5	1.0106	1.0103	1.0093	1.0076	1.0053	1.0023	(0,2.42)
	4.0	1.0081	1.0078	1.0071	1.0058	1.0040	1.0018	(0,2.42)
	4.5	1.0064	1.0062	1.0056	1.0046	1.0032	1.0014	(0,2.42)
	5.0	1.0051	1.0049	1.0044	1.0036	1.0025	1.0011	(0,2.42)

Table 5. The values of e_i for $i = 14$ *and* 15 (cont.)

n	β	$e_{15} = RE\left(T^*_{14}, T_{04}\right)$					Range of λ in which T^*_{14} is efficient to T_{04}
		$\lambda = 1.00$	$\lambda = 1.25$ and $\lambda = 0.75$	$\lambda = 1.50$ and $\lambda = 0.50$	$\lambda = 1.75$ and $\lambda = 0.25$	$\lambda = 2.00$ and $\lambda = 0.00$	
2	2.5	1.2799	1.2578	1.1962	1.1058	1.0000	[0,2]
	3.0	1.1674	1.1553	1.1205	1.0669	1.0000	[0,2]
	3.5	1.1135	1.1056	1.0828	1.0467	1.0000	[0,2]
	4.0	1.0827	1.0772	1.0608	1.0346	1.0000	[0,2]
	4.5	1.0633	1.0591	1.0467	1.0267	1.0000	[0,2]
	5.0	1.0501	1.0469	1.0371	1.0213	1.0000	[0,2]
3	2.5	1.0832	1.0776	1.0611	1.0348	1.0000	[0,2]
	3.0	1.0543	1.0508	1.0402	1.0231	1.0000	[0,2]
	3.5	1.0385	1.0360	1.0286	1.0165	1.0000	[0,2]
	4.0	1.0288	1.0270	1.0215	1.0124	1.0000	[0,2]
	4.5	1.0224	1.0210	1.0167	1.0097	1.0000	[0,2]
	5.0	1.0180	1.0168	1.0134	1.0078	1.0000	[0,2]
4	2.5	1.0477	1.0446	1.0353	1.0203	1.0000	[0,2]
	3.0	1.0317	1.0297	1.0236	1.0136	1.0000	[0,2]
	3.5	1.0227	1.0213	1.0169	1.0098	1.0000	[0,2]
	4.0	1.0171	1.0160	1.0128	1.0074	1.0000	[0,2]
	4.5	1.0134	1.0125	1.0100	1.0058	1.0000	[0,2]
	5.0	1.0107	1.0101	1.0080	1.0047	1.0000	[0,2]
5	2.5	1.0278	1.0260	1.0207	1.0120	1.0000	[0,2]
	3.0	1.0187	1.0175	1.0140	1.0081	1.0000	[0,2]
	3.5	1.0135	1.0127	1.0101	1.0059	1.0000	[0,2]
	4.0	1.0101	1.0095	1.0076	1.0044	1.0000	[0,2]
	4.5	1.0080	1.0075	1.0060	1.0035	1.0000	[0,2]
	5.0	1.0065	1.0061	1.0048	1.0028	1.0000	[0,2]
6	2.5	1.0196	1.0183	1.0146	1.0085	1.0000	[0,2]
	3.0	1.0133	1.0124	1.0099	1.0058	1.0000	[0,2]
	3.5	1.0096	1.0090	1.0072	1.0042	1.0000	[0,2]
	4.0	1.0073	1.0068	1.0055	1.0032	1.0000	[0,2]
	4.5	1.0057	1.0054	1.0043	1.0025	1.0000	[0,2]
	5.0	1.0046	1.0043	1.0035	1.0020	1.0000	[0,2]
7	2.5	1.0138	1.0129	1.0103	1.0060	1.0000	[0,2]
	3.0	1.0094	1.0088	1.0070	1.0041	1.0000	[0,2]
	3.5	1.0068	1.0064	1.0051	1.0030	1.0000	[0,2]
	4.0	1.0052	1.0049	1.0039	1.0023	1.0000	[0,2]
	4.5	1.0041	1.0038	1.0031	1.0018	1.0000	[0,2]
	5.0	1.0033	1.0031	1.0025	1.0014	1.0000	[0,2]
8	2.5	1.0107	1.0100	1.0080	1.0046	1.0000	[0,2]
	3.0	1.0073	1.0068	1.0055	1.0032	1.0000	[0,2]
	3.5	1.0053	1.0050	1.0040	1.0023	1.0000	[0,2]
	4.0	1.0040	1.0038	1.0030	1.0018	1.0000	[0,2]
	4.5	1.0032	1.0030	1.0024	1.0014	1.0000	[0,2]
	5.0	1.0025	1.0024	1.0019	1.0011	1.0000	[0,2]

5. Conclusion

It is observed from Table 1 that the values of the relative efficiencies $e_i's, i = 1,2,...,7$ of the proposed minimum mean squared error (MMSE) estimators $T_{0i}, i = 1,2,3,4$ with respect to Lesitha and Thomas (2012) estimators $t_i's, i = 1,2,3,4$ respectively are greater than 'unity'. Thus, the proposed estimators $T_{0i}, i = 1,2,3,4$ are more efficient than the corresponding usual estimators $t_i's, i = 1,2,3,4$ respectively.

It is further observed that the values of the relative efficiency e_5 of T_{04} with respect to T_{01} are the largest among $e_i's, i = 1,2,...,7$, from which follows that the proposed MMSE estimator T_{04} is the best estimator among Lesitha and Thomas (2012) estimators $t_i's, i = 1,2,3,4$ and MMSE estimators $T_{0i}, i = 1,2,3,4$.

Tables 2 to 5 demonstrate that for fixed (n, β) the values of relative efficiencies $e_i's, i = 8,9,...,15$ increase as λ increases up to 1, while it decreases if λ goes beyond 'unity'. When the value of λ is unity (i.e. the guessed value α_0 coincides with the true value α) a higher gain in efficiency is seen. For fixed values of (n, λ) the values of $e_i's, i = 8,9,...,15$ decrease as β increases. When (β, λ) are fixed the values of $e_i's, i = 8,9,...,15$ also decrease as sample size n increases. A higher gain in efficiency is obtained when the sample size n is small. In general, the estimators $T_{1i}^*, i = 1,2,3,4$ are more efficient than Lesitha and Thomas (2012) estimators $t_i's, i = 1,2,3,4$ and MMSE estimators $T_{0i}, i = 1,2,3,4$ respectively when $\lambda \in (0, 2.42)$. It is further observed that $T_{1i}^*, i = 1,2,3,4$ are respectively better than MMSE estimators $T_{0i}, i = 1,2,3,4$ when $\lambda \in [0,2]$.

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