STATISTICS IN TRANSITION new series, Spring 2014
Vol. 15, No. 2, pp. 317–326

ON CERTAIN A-OPTIMAL BIASED SPRING BALANCE WEIGHING DESIGNS

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ABSTRACT

In the paper, the estimation of unknown measurements of *p* objects in the experiment, according to the model of the spring balance weighing design, is discussed. The weighing design is called biased if the first column of the design matrix has elements equal to one only. The A-optimal design is a design in which the trace of the inverse of information matrix is minimal. The main result is the broadening of the class of experimental designs so that we are able to determine the regular A-optimal design. We give the lowest bound of the covariance matrix of errors and the conditions under which this lowest bound is attained. Moreover, we give new construction methods of theregular A-optimal spring balance weighing design based on the incidence matrices of the balanced incomplete block designs. The example is also given.

Key words: A-optimal design, spring balance weighing design.

1. Introduction

Let us consider $\mathbf{\Phi}_{n\times p}(0,1)$, the class of all possible $n \times p$ matrices of the elements equal to zero or one and, moreover, the first column of this matrix consists only of ones. Any matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ is called the design matrix of the biased spring balance weighing design if the result of the experiment we are able to present in the form $\mathbf{y} = \mathbf{X}\mathbf{w}^* + \mathbf{e}$, where $\mathbf{y}$ is an $n \times 1$ vector of observations, $\mathbf{w}^*$ is a $p \times 1$ vector of unknown parameters and $\mathbf{e}$ is an $n \times 1$ vector of random errors. Furthermore, assume that $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$ and $\mathrm{Var}(\mathbf{e}) = \sigma^2\mathbf{G}$, where $\mathbf{0}_n$ is vector of zeros, σ^2 is the constant variance of errors, $\mathbf{G}$ is the $n \times n$ symmetric positive definite diagonal matrix of known elements.

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The normal equations estimating $\mathbf{w}^*$ are of the form $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\hat{\mathbf{w}}^* = \mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$, where $\hat{\mathbf{w}}^*$ is the vector of the weights estimated by the least squares method. Any weighing design is nonsingular if the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular. It is obvious that $\mathbf{G}$ is the symmetric positive definite matrix, and any weighing design is nonsingular if and only if the matrix $\mathbf{X}'\mathbf{X}$ is nonsingular and then in that case all the parameters are estimable. The estimator of the vector representing unknown measurements of objects $\mathbf{w}^*$ is equal to $\hat{\mathbf{w}}^* = \left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$ assuming that $\mathbf{X}$ is of full column rank. The covariance matrix of $\hat{\mathbf{w}}^*$ is given by $\mathrm{Var}\left(\hat{\mathbf{w}}^*\right) = \sigma^2\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$.

In the special case of experimental designs, when bias is present, then $\mathbf{w}^* = \begin{bmatrix} w_1 & \mathbf{w} \end{bmatrix}'$ is the $p \times 1$ vector of unknown measurements of objects, w_1 is the parameter corresponding to the bias (systematic error), $\mathbf{w} = \begin{bmatrix} w_2 & w_3 & \cdots & w_p \end{bmatrix}'$ is the $(p-1)\times 1$ vector of unknown measurements of object excluding bias. In such experiment we assume that there is one object whose value is estimated by taking the column of ones in the design matrix $\mathbf{X}$ corresponding to the bias. Thus, we consider the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ in the following form

$$\mathbf{X} = \begin{bmatrix} \mathbf{1}_n & \mathbf{X}_1 \end{bmatrix}, \tag{1}$$

where $\mathbf{1}_n$ is $n \times 1$ vector of ones, $\mathbf{X}_1$ is $n \times (p-1)$ matrix of elements equal to zero or one.

It is worth emphasizing that for each pattern of $\mathbf{G}$ the conditions determining optimal design must be separately investigated. For the case $\mathbf{G} = \mathbf{I}_n$, Banerjee (1975), Raghavarao (1971) and Katulska (1989) present the problems related to the biased spring balance weighing designs. Some considerations connected with the diagonal covariance matrix of errors $\sigma^2\mathbf{G}$ are presented in Ceranka and Katulska (1990, 1992).

2. The main result

The statistical problem is to determine the most efficient design in some sense by a proper choice of the design matrix $\mathbf{X}$ among many at our disposals in $\mathbf{\Phi}_{n\times p}(0,1)$. Some optimal criteria have been considered in the literature, see Pukelsheim (1993). One of them is A-optimality which minimizes the average variance of the estimator of unknown measurements of the objects.

For the case $\mathbf{G}$ there is a positive definite diagonal matrix of known elements. The problems related to the regular A-optimal biased spring balance weighing design have been considered in the literature, see, for instance, Graczyk (2011). In this paper, the following definition is presented.

Definition 1. Any nonsingular $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ of the form (1) with the diagonal covariance matrix of errors $\sigma^2\mathbf{G}$ is called the regular A-optimal biased spring balance weighing design for estimation of $\hat{\mathbf{w}}$ if $\mathrm{tr}(\mathrm{Var}(\hat{\mathbf{w}})) = \sigma^2 \dfrac{4(p-1)}{\mathrm{tr}(\mathbf{G}^{-1})}$.

In addition, in the same paper, the following corollaries are presented.

Corollary 1. Any nonsingular $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ of the form (1) with the diagonal covariance matrix of errors $\sigma^2\mathbf{G}$ is called the regular A-optimal biased spring balance weighing design for estimation of $\hat{\mathbf{w}}$ if and only if $\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1 = \dfrac{\mathrm{tr}(\mathbf{G}^{-1})}{4}\left(\mathbf{I}_{p-1} + \mathbf{1}_{p-1}\mathbf{1}_{p-1}'\right)$.

Corollary 2. In the regular A-optimal biased spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ of the form (1) with the diagonal covariance matrix of errors $\sigma^2\mathbf{G}$, $\mathrm{Var}(\hat{w}_1) = \dfrac{p\sigma^2}{\mathrm{tr}(G^{-1})}$, where $\hat{w}_1$ is the estimator of the bias.

In the present paper, we construct the regular A-optimal biased spring balance weighing design with the covariance matrix of errors $\sigma^2\mathbf{G}$ for $\mathbf{G}$ of the form

$$\mathbf{G} = \begin{bmatrix} g^{-1}\mathbf{I}_c & \mathbf{0}_c\mathbf{0}_d' & \mathbf{0}_c\mathbf{0}_{n-c-d}' \\ \mathbf{0}_d\mathbf{0}_c' & g_1^{-1}\mathbf{I}_d & \mathbf{0}_d\mathbf{0}_{n-c-d}' \\ \mathbf{0}_{n-c-d}\mathbf{0}_c' & \mathbf{0}_{n-c-d}\mathbf{0}_d' & \mathbf{I}_{n-c-d} \end{bmatrix}, \; g, g_1 > 0, c, d \geq 0. \quad (2)$$

Suppose further that the design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ is partitioned in the same way as the matrix $\mathbf{G}$, i.e. we have

$$\mathbf{X} = \begin{bmatrix} \mathbf{1}_c & \mathbf{0}_c\mathbf{0}_{p-1}' \\ \mathbf{1}_d & \mathbf{1}_d\mathbf{1}_{p-1}' \\ \mathbf{1}_{n-c-d} & \mathbf{X}_2 \end{bmatrix}, \; c, d \geq 0. \quad (3)$$

In the special case $c = 0$ (or $d = 0$), the respective element of the matrix does not exist. That way we obtain the Theorem.

Theorem 1. Any nonsingular biased spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ given by (3) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (2), is the regular A-optimal if and only if

$$\mathbf{X}_2^{'}\mathbf{X}_2 = \frac{n + c(g-1) + d(g_1 - 1)}{4}\mathbf{I}_{p-1} + \frac{n + c(g-1) - d(3g_1 + 1)}{4}\mathbf{1}_{p-1}\mathbf{1}_{p-1}^{'}, \qquad (4)$$

$g, g_1 > 0, \ c, d \geq 0$.

Proof. If $\mathbf{X}_1 = \left[\mathbf{0}_{p-1}\mathbf{0}_c^{'} \quad \mathbf{1}_{p-1}^{'}\mathbf{1}_d \quad \mathbf{X}_2^{'}\right]^{'}$ then $\mathbf{X}_1^{'}\mathbf{G}^{-1}\mathbf{X}_1 = g_1 d\mathbf{1}_{p-1}\mathbf{1}_{p-1}^{'} + \mathbf{X}_2^{'}\mathbf{X}_2$. Consequently, in that case $\mathrm{tr}\left(\mathbf{G}^{-1}\right) = n + c(g-1) + d(g_1 - 1)$. From the above and from Corollary 1, $\mathbf{X}_1^{'}\mathbf{G}^{-1}\mathbf{X}_1 = \frac{n + c(g-1) + d(g_1 - 1)}{4}\left(\mathbf{I}_{p-1} + \mathbf{1}_{p-1}\mathbf{1}_{p-1}^{'}\right)$. This gives

$$\mathbf{X}_2^{'}\mathbf{X}_2 = \frac{n + c(g-1) + d(g_1 - 1)}{4}\mathbf{I}_{p-1} + \frac{n + c(g-1) - d(3g_1 + 1)}{4}\mathbf{1}_{p-1}\mathbf{1}_{p-1}^{'},$$

when $g, g_1 > 0, \ c, d \geq 0$. Hence the Theorem follows.

It is worth noting that the condition (4) implies that for the matrix $\mathbf{X}_2^{'}\mathbf{X}_2$, diagonal elements satisfy the condition $n + c(g-1) - d(g_1 + 1) \equiv 0 \bmod(2)$ and off-diagonal elements satisfy the condition $n + c(g-1) + d(g_1 - 1) \equiv 0 \bmod(4)$. Afterwards, we have to determine the matrix $\mathbf{X}_2$ of elements equal to 1 or 0 which satisfied these conditions. Several methods of construction of the design matrix of the optimal spring balance weighing design are presented in the literature. Some of them are based on the incidence matrices of known block designs, another ones rely on using some algorithms.

In the following part of the paper we present the problem of constructing a regular A-optimal biased spring balance weighing design based on the incidence matrix of the balanced incomplete block design.

3. Regular A-optimal designs

Here, we present the application of the incidence matrix of balanced incomplete block design to the construction of the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ of the regular A-optimal spring balance weighing design. The balanced incomplete block design with the parameters v, b, r, k, λ is the design where we replace v objects in b blocks, each of size k. That is why each object occurs r times altogether and each pair of different objects occurs together in λ blocks. For more details see Raghavarao and Padgett (2005). Let us denote by $\mathbf{N}$ the

$v \times b$ incidence matrix of the binary incomplete block design. Then the matrix $\mathbf{X}$ is shown by the equations

$$\mathbf{X} = \begin{bmatrix} \mathbf{1}_c & \mathbf{0}_c \mathbf{0}_v^{'} \\ \mathbf{1}_d & \mathbf{1}_d \mathbf{1}_v^{'} \\ \mathbf{1}_b & \mathbf{N}^{'} \end{bmatrix}, \; c, d \geq 0 . \tag{5}$$

$\mathbf{X} \in \mathbf{\Phi}_{n \times p}(0,1)$ in the form (5) is the matrix of the biased spring balance weighing design. In this design we determine unknown measurements of $p = v + 1$ in $n = b + c + d$ measurement operations.

Theorem 2. The biased spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(0,1)$ given by (5) with the covariance matrix of errors $\sigma^2 \mathbf{G}$, where $\mathbf{G}$ is given by (2) is the regular A-optimal if and only if

$$b + cg - 3dg_1 \equiv 0 \bmod(4) \tag{6}$$

and

$$r = 2\lambda + dg_1 . \tag{7}$$

Proof. The main idea of the proof is to show that any biased spring balance weighing design $\mathbf{X}$ given by (5) with the covariance matrix of errors $\sigma^2 \mathbf{G}$, where $\mathbf{G}$ is given by (2), is the regular A-optimal if and only if $\mathbf{NN}^{'} = \frac{b + cg + dg_1}{4} \mathbf{I}_v + \frac{b + cg - 3dg_1}{4} \mathbf{1}_v \mathbf{1}_v^{'}$, $c, d \geq 0$, which follows from Theorem 1. Assume the formula $\mathbf{NN}^{'} = (r - \lambda)\mathbf{I}_v + \lambda \mathbf{1}_v \mathbf{1}_v^{'}$ holds, then we obtain the equalities (6) and (7) that is our claim.

Theorem 3. If there exists a balanced incomplete block design with the parameters v, b, r, k, λ and the design matrix $\mathbf{X}$ given by (5) is the matrix of the regular A-optimal biased spring balance weighing design with the covariance matrix of errors $\sigma^2 \mathbf{G}$, where $\mathbf{G}$ is given by (2), then

$$\begin{aligned} v &= \frac{(4\lambda + 3dg_1 - cg)(\lambda + dg_1)}{dg_1 \lambda + cg\lambda + d^2 g_1^2} \\ b &= 4\lambda + 3dg_1 - cg \\ r &= 2\lambda + dg_1 \\ k &= \frac{(2\lambda + dg_1)(\lambda + dg_1)}{dg_1 \lambda + cg\lambda + d^2 g_1^2} \end{aligned} . \tag{8}$$

Proof. Let us first observe that from (6) and (7) it follows that $r = 2\lambda + dg_1$ and $b = 4\lambda + 3dg_1 - cg$. The proof is completed by showing that if the parameters v, b, r, k, λ of the balanced incomplete block design satisfy the conditions $vr = bk$ and $\lambda(v-1) = r(k-1)$, then v and k are given as in (8).

We have seen in Theorem 3 that if the parameters of the balanced incomplete block design satisfy the condition (8) then the biased spring balance weighing design $\mathbf{X}$ given by formula (5) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (2), is the regular A-optimal. The parameters v, b, r, k, λ must be positive integers as the parameters of the balanced incomplete block design. From the above reasoning and the condition (8) we obtain the theorem.

Theorem 4. For any positive definite integer λ and integers $c, d \geq 0$ the parameters v, b, r, k given by (8) are positive integers if and only if one of the following conditions holds:

(i) $d = 0$, $c > 0$, $2\lambda \equiv 0 \bmod(cg)$,

(ii) $d > 0$, $c = 0$, $2\lambda \equiv 0 \bmod(dg_1)$,

(iii) $d > 0$, $c = \dfrac{2(\lambda + dg_1)}{g}$, $2(\lambda + dg_1) \equiv 0 \bmod(g)$,

(iv) $dg_1 = cg$, $\lambda \equiv 0 \bmod(dg_1)$.

Proof. It is sufficient to show that from the condition (8) it follows that $v = 2k + a$, where $a = \dfrac{(dg_1 - cg)(\lambda + dg_1)}{dg_1\lambda + cg\lambda + d^2 g_1^2}$ is such an integer that v is a positive integer. If $d = 0$, $c > 0$ then $a = -1$ and $k = \dfrac{2\lambda}{cg}$. It implies that $2\lambda \equiv 0 \bmod(cg)$, i.e. the condition (i) is fulfilled. By similar arguments, if $d > 0$, $c = 0$ then $a = 1$ and $k = 1 + \dfrac{2\lambda}{dg_1}$, and it implies that $2\lambda \equiv 0 \bmod(dg_1)$, i.e. the condition (ii) holds. If $d > 0$, $c = \dfrac{2(\lambda + dg_1)}{g}$, then $2(\lambda + dg_1) \equiv 0 \bmod(g)$, $a = -1$, $k = 1$, which means the condition (iii) is true. If $dg_1 = cg$ then $a = 0$ and $k = 1 + \dfrac{\lambda}{dg_1}$, it implies $\lambda \equiv 0 \bmod(dg_1)$. If one of the conditions (i)-(iv) is fulfilled then it is obvious that v, b, r, k are positive integers, which is the desired conclusion.

Based on the theoretical results presented in Theorems 3 and 4 we can formulate the following Corollaries.

Corollary 3. If there exists a balanced incomplete block design with the parameters v, b, r, k, λ and the matrix $\mathbf{X}$ given by (5) is the matrix of the regular A-optimal biased spring balance weighing design with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (2), then

(i) $v = 2k - 1$, $b = 4\lambda - cg$, $r = 2\lambda$, $k = \dfrac{2\lambda}{cg}$ if $c > 0$ and $2\lambda \equiv 0 \bmod(cg)$,

(ii) $v = 2k + 1$, $b = 4\lambda + 3dg_1$, $r = 2\lambda + dg_1$, $k = 1 + \dfrac{2\lambda}{dg_1}$ if $d > 0$ and $2\lambda \equiv 0 \bmod(dg_1)$,

(iii) $v = 2k$, $b = 2(2\lambda + dg_1)$, $r = 2\lambda + dg_1$, $k = 1 + \dfrac{\lambda}{dg_1}$ if $d > 0$ and $\lambda \equiv 0 \bmod(dg_1)$.

Corollary 4. If $d > 0$, $c = 2g^{-1}(\lambda + dg_1)$, $2(\lambda + dg_1) \equiv 0 \bmod(g)$, then $v = k = 1$, $b = r = 2\lambda + dg_1$ and $\mathbf{X} = \begin{bmatrix} \mathbf{1}_c & \mathbf{0}_c \\ \mathbf{1}_d & \mathbf{1}_d \\ \mathbf{1}_b & \mathbf{1}_b \end{bmatrix}$ is the matrix of the regular A-optimal biased spring balance weighing design with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (2), for two objects.

We have seen in Corollary 3 that if the parameters of the balanced incomplete block design are of the form (i)-(iii), then a biased spring balance weighing design $\mathbf{X}$, given by (5) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (2), is the regular A-optimal. Then we obtain the series of the parameters of the balanced incomplete block designs. Based on these parameters we form the incidence matrix $\mathbf{N}$ and then the design matrix $\mathbf{X}$.

Corollary 5. Let $d = 0$ and let $\mathbf{N}$ be the incidence matrix of the balanced incomplete block design with the parameters

(i) $v = 4t - 1$, $b = cg(4t - 1)$, $r = 2cgt$, $k = 2t$, $\lambda = cgt$, $t = 1,2,\ldots$, for odd cg,

(ii) $v = 2t - 1$, $b = cg(2t - 1)$, $r = cgt$, $k = t$, $\lambda = \dfrac{cgt}{2}$, $t = 2,3,\ldots$, for even cg,

then the matrix $\mathbf{X}$ given by (5) is the regular A-optimal biased spring balance weighing design with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (2).

Corollary 6. Let $c = 0$ and let $\mathbf{N}$ be the incidence matrix of the balanced incomplete block design with the parameters

(i) $v = 4t + 3$, $b = dg_1(4t + 3)$, $r = dg_1(2t + 1)$, $k = 2t + 1$, $\lambda = dg_1 t$ for odd dg_1,

(ii) $v = 2t + 1$, $b = dg_1(2t + 1)$, $r = dg_1(t + 1)$, $k = t + 1$, $\lambda = \frac{dg_1 t}{2}$ for even dg_1,

$t = 1,2,\ldots$, then the matrix $\mathbf{X}$ given by (5) is the regular A-optimal biased spring balance weighing design with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (2).

Corollary 7. Let $dg_1 = cg$ and let $\mathbf{N}$ be the incidence matrix of the balanced incomplete block design with the parameters $v = 2(t + 1)$, $b = 2dg_1(2t + 1)$, $r = dg_1(2t + 1)$, $k = t + 1$, $\lambda = dg_1 t$, $t = 1,2,\ldots$, then the matrix $\mathbf{X}$ given by (5) is the regular A-optimal biased spring balance weighing design with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (2).

4. Example

Weighing designs can be applied in all experiments in which the experimental factors are on two levels. Let us suppose we study four marketing factors: the kind of advertisement (television or outdoor), assortment (basic or complementary), personal promotion (present or not), sale (directly by producer, mail-order). We will study the level of sale of chosen product on the basis on the nationwide range. From the statistical point of view, we are interested in determining the influences of these factors using sixteen different combinations. In the notation of weighing designs we determine unknown measurements of $p = 5$ objects in $n = 16$ measurements, so $\mathbf{X} \in \mathbf{\Phi}_{16\times 5}(0,1)$. In order to illustrate the theory given above we consider the case that we compare the influence of these factors in three different cities. Thus, the variance matrix of errors $\sigma^2\mathbf{G}$ is given by the matrix
$$\mathbf{G} = \begin{bmatrix} \frac{1}{2} & \mathbf{0}_3' & \mathbf{0}_{12}' \\ \mathbf{0}_{12} & \frac{3}{2}\mathbf{I}_3 & \mathbf{0}_3\mathbf{0}_{12}' \\ \mathbf{0}_{12} & \mathbf{0}_{12}\mathbf{0}_3' & \mathbf{I}_{12} \end{bmatrix}$$
for $c = 1, d = 3, g = 2, g_1 = \frac{2}{3}$.

Moreover, $\mathrm{tr}(\mathbf{G}^{-1}) = 16$. Then we form the design matrix $\mathbf{X}$ of the form (5) for the case $c, d > 0$. That is why we consider $\mathbf{N} = \mathbf{1}_2 \otimes \mathbf{N}_1$, where $\mathbf{N}_1$ is the incidence matrix of the balanced incomplete block design with the parameters

$v=4$, $b=6$, $r=2$, $k=3$, $\lambda=1$ given by $\mathbf{N}_1=\begin{bmatrix}1&1&1&0&0&0\\1&0&0&1&1&0\\0&1&0&1&0&1\\0&0&1&0&1&1\end{bmatrix}$. Thus we obtain

$$\mathbf{X}=\begin{bmatrix}1&\mathbf{0}_4'\\\mathbf{1}_3&\mathbf{1}_3\mathbf{1}_4'\\\mathbf{1}_{12}&\mathbf{N}'\end{bmatrix}=\begin{bmatrix}1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\0&1&1&1&1&1&1&0&0&0&1&1&1&0&0&0\\0&1&1&1&1&0&0&1&1&0&1&0&0&1&1&0\\0&1&1&1&0&1&0&1&0&1&0&1&0&1&0&1\\0&1&1&1&0&0&1&0&1&1&0&0&1&0&1&1\end{bmatrix}'$$ and

$\mathbf{X}_1'\mathbf{G}^{-1}\mathbf{X}_1=4\left[\mathbf{I}_4+\mathbf{1}_4\mathbf{1}_4'\right]$, i.e. the design $\mathbf{X}$ is the regular A-optimal (see Corollary 1). The first column of the design matrix $\mathbf{X}$ responds to the influence of a nationwide range, the second one to the kind of advertisement, the third one to the influence of the assortment. The next column exposes a personal promotion, and finally the kind of sale. The form of the matrix can be interpreted in the following sense: the eighth row indicates that we take the nationwide range, a basic assortment and a personal promotion. Let us suppose $\mathbf{y}$ be the 16×1 vector of the results of the experiment. Thus $\hat{\mathbf{w}}^*=\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}=$

$$\frac{1}{24}\begin{bmatrix}15&-3&-3&-3&15&15&15&15&15&15&15&15&15&15&15&15\\-6&2&2&2&3&3&3&3&3&3&3&3&3&3&3&3\\-6&2&2&2&3&3&3&3&3&3&3&3&3&3&3&3\\-6&2&2&2&3&3&3&3&3&3&3&3&3&3&3&3\\-6&2&2&2&3&3&3&3&3&3&3&3&3&3&3&3\end{bmatrix}\mathbf{y}.$$

5. Discussion

In the paper, some problems related to A-optimality criterion are presented. The special class of experimental designs, i.e. biased spring balance weighing designs are considered here. It is not possible to determine a regular A-optimal biased spring balance weighing design in any class $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$. Therefore, in the literature new construction methods of A-optimal designs have been presented. In the most cases the construction of such design is based on the incidence matrices of some known block designs. It is worth emphasizing that in the regular A-optimal biased spring balance weighing design we are able to determine unknown measurements of the object with a minimal average variance. From the viewpoint of the experimenters such property is expectable.

It is clear that in the case presented in the example, the experimental design 2^4 may be used. It should be underlined that, for $\mathbf{G} = \mathbf{I}_n$, the sum of variances of estimators of the vector of unknown parameters in both designs is the same. When $\mathbf{G}$ is any positive definite diagonal matrix, the sum of variances of estimators of the vector of unknown parameters in the regular A-optimal spring balance weighing design is less than the sum of variances of estimators of the vector of unknown parameters in the design 2^4.

REFERENCES

BANERJEE, K. S., (1975). Weighing Designs for Chemistry, Medicine, Economics, Operations Research, Statistics. Marcel Dekker Inc., New York.

CERANKA, B., KATULSKA, K., (1990). Constructions of optimum biased spring balance weighing designs with diagonal covariance matrix of errors. Computational Statistics and Data Analysis 10, pp. 121–131.

CERANKA, B., KATULSKA K., (1992). Optimum biased spring balance weighing designs with non-homogeneity of the variances of errors. Journal of Statistical Planning and Inference 30, pp. 185–193.

GRACZYK, M., (2011). A-optimal biased spring balance weighing design. Kybernetika 47, pp. 893–901.

KATULSKA, K., (1989). Optimum biased spring balance weighing designs. Statistics and Probability Letters 8, pp. 267–271.

PUKELSHEIM, F., (1993). Optimal design of experiment. John Wiley and Sons. New York.

RAGHAVARAO, D., (1971). Constructions and combinatorial problems in design of experiment. John Wiley and Sons. New York.

RAGHAVARAO, D., PADGETT, L. V., (2005). Block Designs, Analysis, Combinatorics and Applications. Series of Applied Mathematics 17, Word Scientific Publishing Co. Pte. Ltd., Singapore.